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NOTE II

ON THE *HEXAGRAMMUM MYSTICUM* OF PASCAL

If we have in a plane two triads of points, U, V, W and U', V', W', which are in perspective, the joins UU', VV', WW' meeting in a point, and the intersections of the pairs of lines, VW and $V'W'$, WU and $W'U'$, UV and $U'V'$ being in line, then the remaining intersections of the lines VW, WU, UV with the lines $V'W'$, $W'U'$, $U'V'$ are six points which lie on a conic, as we know from what we have in the text called the converse of Pascal's theorem; and, if we take these six points in any order, the triad formed by one set of alternate joins of these six points is in perspective with the triad formed by the other set of alternate joins, as we know from Pascal's theorem. The axes of perspective of these various pairs of triads are called Pascal lines; they are sixty in number. They cointersect in sets of three (or of four) in various ways, and these points of intersection lie in sets upon lines, as has gradually appeared from the work of many mathematicians, Steiner, Kirkman, Cayley, Salmon, Veronese, Cremona, and others. The proof of the properties in the plane, though interesting, is intricate. It depends upon successive applications of Desargues' theorem, which we have learnt to regard as deduced from Propositions of Incidence in three dimensions; and it is the fact that all the results which have elicited attention are such as may be obtained by projection from a figure existing in such a space; this manner of proof is much simpler than the proof in the plane. Applied by Cayley in a particular way (*Coll. Papers*, VI, pp. 129–134 (1868)), it was applied by Cremona to a figure more general than Pascal's (*Memorie d. r. Acc. d. Lincei*, I, 1877, pp. 854–874). Three distinct points of Cremona's treatment of the figure in three dimensions are, (*a*) That it depends upon a configuration of fifteen lines lying in threes in fifteen planes, of which three pass through each line, (*b*) That this configuration is deducible from a figure of six planes, forming, in Cremona's phrase, the *nocciolo* of the whole, (*c*) That when the equations of these planes are introduced, only one of the identical relations connecting these comes into consideration. It follows from the last that we may regard the six loci as being in a space of four dimensions. This procedure, which Richmond has used, leads to a very simple treatment of the whole matter, which is the more interesting because the figure in four dimensions is one which, as we shall see in a subsequent volume, is of fundamental importance. Cremona's figure of fifteen lines lying by threes in fifteen planes is an example of the (3, 3) correspondence which it is possible to set up (in various ways) between any two sets of fifteen things; this had been considered by Sylvester, in connexion with the problem of naming a function of six letters capable only of six values under permutation of the letters (Sylvester, *Coll. Papers*, I, p. 92 (1844); II, p. 265 (1861)); it is pertinent and instructive for the problem in hand to consider this. In the following note we have (1) dealt with this problem of arrangements, (2) shewn how this can be represented by figures in four, three and two dimensions, (3) pointed out the properties of the figures which lead to the classical properties of the Pascal figure, (4) explained the exact figure, relating to a cubic surface with a node, by which Cremona was led to the generalisation, (5) given some examples of the proof in the plane of the properties of Pascal's figure, and of the configurations arising from some selected portions of the figure in three dimensions, and (6) made reference to the more important original authorities. The reader may prefer to read (4) before the other sections.

(1) Consider six elements, which we denote, at present, by 1, 2, 3, 4, 5, 6. These can be arranged in three pairs, for example 12, 34, 56, wherein, in each pair, the order of the elements is indifferent, and the order of the pairs, in each such set of three pairs, is also indifferent. Such a set of three pairs, involving all the elements, is what was called by Sylvester a *syntheme*, each of the pairs being what he called a *duad*. The total number of possible duads is fifteen; this is also the total number of possible synthemes. It is possible to choose a set of five synthemes which contain, in their aggregate, all the fifteen duads. Such a set of synthemes is called a system. The total number of possible systems is six; if these be all taken, each containing five synthemes, every syntheme will occur twice, in different systems; as has been said, each duad occurs once in each system. Such an arrangement was given by Sylvester (see below). If we assign names to the systems, say P, Q, R, P', Q', R', each syntheme, as occurring in two systems, will correspond to two of these six letters. Thus every one of the fifteen pairs which can be formed by two of these six letters, say, a letter-duad, will correspond to three number-duads, chosen from the possible fifteen duads of numbers, forming a syntheme. The converse is also true; any duad of the numbers being taken, the remaining four numbers can be taken in pairs in three ways; thus any duad occurs in three synthemes, and each of these synthemes enters in two of the systems. The duad thus corresponds to three pairs of systems; or any one of the fifteen number-duads corresponds to three letter-duads, chosen from the possible duads of P, Q, R, P', Q', R'. Therefore, as any

fifteen things can be identified by naming them after the pairs which can be formed from six arbitrary symbols, we can have a (3, 3) correspondence between any two sets of fifteen things. As has been said, the importance of this for the present purpose was emphasized by Cremona (*loc. cit.* pp. 854, 866, 870). A further remark which is of use is that two synthemes which have no duad in common occur together in only one system, since two systems have only one syntheme in common. Two such synthemes thus serve to identify a system. [Add.]

Of the various possible ways of assigning the names $P, Q, \ldots, R'$ to the systems, one example is given in the following scheme, which can be read either in rows or columns.

	P	Q	R	P'	Q'	R'
P		14.25.36	16.24.35	13.26.45	12.34.56	15.23.46
Q	14.25.36		15.26.34	12.35.46	16.23.45	13.24.56
R	16.24.35	15.26.34		14.23.56	13.25.46	12.36.45
P'	13.26.45	12.35.46	14.23.56		15.24.36	16.25.34
Q'	12.34.56	16.23.45	13.25.46	15.24.36		14.26.35
R'	15.23.46	13.24.56	12.36.45	16.25.34	14.26.35	

With this table, the correspondence of the pairs of systems with the synthemes can be enumerated at once; for instance, (Q, R) is associated with (15.26.34), and (R, P') with (14.23.56). Conversely, the duads of the numbers are each associated with three duads of letters; for example, (14) with $(PQ\,.\,RP'\,.\,Q'R')$, since (14) occurs in the same syntheme in P and Q, in the same, other, syntheme in R and P', and in the same, still other, syntheme in Q' and R'. This set of three pairs of letters forms a syntheme, and the synthemes of the letters can similarly be arranged in six systems, with the names 1, 2, 3, 4, 5, 6.

We shall also denote the number-synthemes by the letters $a, b, c, d, a', b', c', d', e, l, m, n, p, q, r$, by the rule which is found by identifying the preceding scheme with the scheme subjoined:

	P	Q	R	P'	Q'	R'
P		e	d	a	b	c
Q	e		d'	a'	b'	c'
R	d	d'		l	m	n
P'	a	a'	l		r	q
Q'	b	b'	m	r		p
R'	c	c'	n	q	p	

Perhaps the easiest way to describe these notations is by the diagram given below in (2), p. 225, or, in another form, in the Frontispiece of the volume.

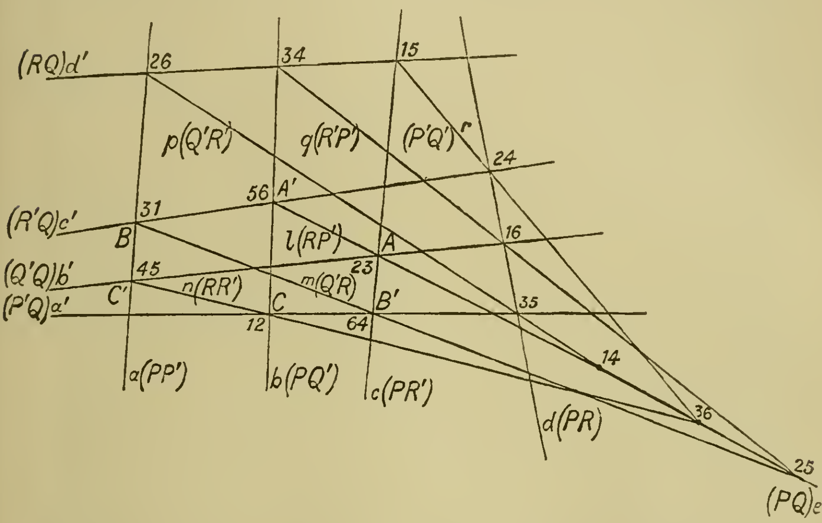
Now let us consider four of the six systems, considered as two pairs, say, for definiteness Q, R' and Q', R. The synthemes common to the two pairs, that is the syntheme common to Q and R', and the syntheme common to Q' and R, will necessarily have a duad in common. Further this aggregate of two letter-duads, Q, R' and Q', R, may be identified by the common number-duad of the two synthemes, taken with one duad from the first syntheme, and one duad from the second syntheme, so chosen as to have one of its numbers the same as one of the numbers of its companion duad from the first syntheme. In the particular case chosen, Q and R' have common the syntheme 13 . 24 . 56, while Q' and R have common the syntheme 13 . 25 . 46. These synthemes have the duad 13 common; we say that the combination $QR' \,.\, Q'R$ may be identified by the symbol 13 . 24 . 25, or by 13 . 42 . 46, or by 13 . 56 . 52, or by 13.65.64. All this is quite easy to see. For first, two synthemes, that have two duads identical, are themselves identical; and second, two synthemes that have no duad in common both occur in the same system, and nowhere else, as we have remarked. The two synthemes chosen, the former common to one pair of systems, the latter common to another pair, must therefore have a duad common. Using $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ to denote the numbers $1, 2, \ldots, 6$, in some order, let these synthemes be $\beta'\gamma' \,.\, \alpha\beta \,.\, \gamma\alpha'$ and $\beta'\gamma' \,.\, \alpha\gamma \,.\, \beta\alpha'$. From these we can form the symbol $\beta'\gamma' \,.\, \alpha\beta \,.\, \alpha\gamma$ (as well as three others). Conversely, if this be given, the formation of the two synthemes having $\beta'\gamma'$ in common and containing, respectively, also $\alpha\beta$ and $\alpha\gamma$, is without ambiguity; so that the symbol identifies the two pairs of systems from which it was formed. Such a symbol, or any one of the other three symbols which identify the same pair of systems, we speak of as defining a T-element. The total number of T-elements is thus $\frac{1}{4} \,.\, 15 \,.\, 4 \,.\, 3$, or forty-five. Now take the six systems in any particular order, say, $PQ'RP'QR'$, where it is to be understood that this is considered equivalent with its reverse order, $R'QP'RQ'P$, and may be named beginning with any one of its letters, as for instance by $RP'QR'PQ'$; thus the total number of orders is $(6!)/2 \,.\, 6$, or sixty. Then take, with each pair of letters in this order, that pair which is not contiguous with it on either side; thus, with QR' take $Q'R$, with RP' take $R'P$, and with $P'Q$ take PQ'. For each of these three sets of two pairs, form the corresponding T-element. It will be found that these can be represented by symbols of the respective forms $\beta'\gamma' \,.\, \alpha\beta \,.\, \alpha\gamma$, $\gamma'\alpha' \,.\, \alpha\beta \,.\, \alpha\gamma$, $\alpha'\beta' \,.\, \alpha\beta \,.\, \alpha\gamma$. For instance, with the order $PQ'RP'QR'$, the two pairs $QR', Q'R$ give a symbol 13 . 64 . 65, the two pairs $RP', R'P$ give a symbol 23 . 64 . 65, and the two pairs

$PQ', P'Q$ give a symbol 12 . 64 . 65. And these three symbols have in common the duads 64 . 65, which have the number 6 in common. Conversely consider the aggregate of two duads $\alpha\beta \,.\, \alpha\gamma$, having the number α in common; we shew that this leads to a particular order of the symbols $P, Q, \ldots, R'$. This aggregate belongs to only three symbols of T-elements, namely $\beta'\gamma' \,.\, \alpha\beta \,.\, \alpha\gamma$, $\gamma'\alpha' \,.\, \alpha\beta \,.\, \alpha\gamma$ and $\alpha'\beta' \,.\, \alpha\beta \,.\, \alpha\gamma$. Of these the first arises by combining the two synthemes $\beta'\gamma' \,.\, \alpha\beta \,.\, \gamma\alpha'$, $\beta'\gamma' \,.\, \alpha\gamma \,.\, \beta\alpha'$, and there is no ambiguity as to these; for the moment let these synthemes be denoted, respectively, by p and p'. The symbol $\gamma'\alpha' \,.\, \alpha\beta \,.\, \alpha\gamma$ similarly arises from the two synthemes $\gamma'\alpha' \,.\, \alpha\beta \,.\, \gamma\beta'$, $\gamma'\alpha' \,.\, \alpha\gamma \,.\, \beta\beta'$, which we denote, respectively, by q and q'. Lastly, the symbol $\alpha'\beta' \,.\, \alpha\beta \,.\, \alpha\gamma$ arises from the two synthemes $\alpha'\beta' \,.\, \alpha\beta \,.\, \gamma\gamma'$, $\alpha'\beta' \,.\, \alpha\gamma \,.\, \beta\gamma'$, which we denote, respectively, by r and r'. We have, however, remarked that two synthemes, which have no duad in common, determine a particular system. Thus the synthemes q and r', namely $\gamma'\alpha' \,.\, \alpha\beta \,.\, \gamma\beta'$ and $\alpha'\beta' \,.\, \alpha\gamma \,.\, \beta\gamma'$, determine one of the systems $P, Q, \ldots, R'$; this system we may, for a moment, denote by (qr'). It is easy to see that the whole of the six systems are thus determined by such pairs of the six synthemes, which we may arrange in the order $pq' \,.\, q'r \,.\, rp' \,.\, p'q \,.\, qr' \,.\, r'p$, there being no other pairs of these synthemes which have no duad in common. So arranged, every consecutive pair of these systems determines, with the pair which is not contiguous with it on either side, a T-element. For instance, if we take the systems $pq', q'r, p'q, qr'$, the first pair have common the syntheme q' or $\gamma'\alpha' \,.\, \alpha\gamma \,.\, \beta\beta'$, the second pair have common the syntheme q or $\gamma'\alpha' \,.\, \alpha\beta \,.\, \gamma\beta'$, and these together determine the T-element represented by $\gamma'\alpha' \,.\, \alpha\beta \,.\, \alpha\gamma$.

We see then that the operation which we carry out when, in Pascal's figure, we form the Pascal line for six points $P, Q, \ldots, R'$ of a conic, taken in a particular order, can be carried out exactly with the symbols, the systems of synthemes taking the places of the points of the conic. The join of any two of the six Pascal points is replaced by one of the fifteen number-synthemes; the intersection of two of these joins is replaced by a T-element, represented by such a symbol as $\beta'\gamma' \,.\, \alpha\beta \,.\, \alpha\gamma$; and the Pascal line containing three of these intersections is replaced by a symbol, such as $\alpha\beta \,.\, \alpha\gamma$, consisting of two duads having a number (α) in common. We have seen that any such symbol leads back to a particular order of the six letters $P, Q, \ldots, R'$. It is in accordance with this, that the number of possible symbols $\alpha\beta \,.\, \alpha\gamma$ is 6 . 10 or sixty. It is at once seen in the Pascal figure that through the intersection of two opposite sides of the hexagon there pass four Pascal lines, the two pairs $QR', Q'R$, for example, being non-contiguous in each of the four hexagons $PQ'RP'QR'$, $PQ'RP'R'Q$, $PRQ'P'QR'$, $PRQ'P'R'Q$; this corresponds to the fact remarked that the T-element $\beta'\gamma' \,.\, \alpha\beta \,.\, \alpha\gamma$ is capable also of the representations $\beta'\gamma'.\gamma\alpha'.\alpha\gamma$, $\beta'\gamma'.\alpha\beta.\beta\alpha'$, $\beta'\gamma'.\alpha'\gamma.\alpha'\beta$. The total intersections of the fifteen sides in the Pascal hexagon, in number $\frac{1}{2}15.14$ or 105, consist in fact of ten intersections at each of the six vertices $P, Q, \ldots, R'$, together with an intersection at each of forty-five T-points.

(2) We now consider a geometrical interpretation of the relations we have described. First, in four dimensions, we can set up a figure of fifteen points lying in threes on fifteen lines, of which three pass through each of the points, beginning in various ways. If we take six general points, say F, G, H, R, S, T, of which the symbols will be subject to one relation, which we write in the form $F+G+H+R+S+T=0$, there will be fifteen points of symbols each the sum of two of $F, G, \ldots, T$, and, for instance, the points $F+G$, $H+R$, $S+T$ will be in line; of such lines there will be fifteen. Three of these will pass through each point; for instance, the lines $(F+G, H+R, S+T)$, $(F+G, H+S, R+T)$, $(F+G, H+T, R+S)$ pass through the point $F+G$. Denoting the six points $F, G, \ldots, T$ by 1, 2, ..., 6, each duad, such as 12, may be supposed to represent one of these points, and each syntheme, such as 12.34.56, to represent one of these lines. Or, we may take four arbitrary lines of general position, say, a, b, c, d; every three of these will have, in four dimensions, a single transversal; let the transversal of a, b, c be denoted by d', that of b, c, d by a', and so on, the four transversals being a', b', c', d'. It is then easy to shew that the line, say n, joining the points (a, b'), (a', b) intersects the line, say r, joining the points (c, d'), (c', d); so, the line joining the points (c, a'), (c', a), say m, meets the line joining the points (b, d'), (b', d), say q; and likewise the line joining the points (b, c'), (b', c), say l, meets the line joining the points (a, d'), (a', d), say p. And further that these three points of intersection are in a line, say e. Six fundamental points $F, G, \ldots, T$, from which these statements can be justified, are obtained by regarding the points (b', c), (c', a), (a', b), (b, c'), (c, a'), (a, b') as being, respectively, the points 23, 31, 12, 56, 64, 45. These six points may in fact be taken arbitrarily to obtain such a figure; denoting them, respectively, by A, B, C, A', B', C', the six consecutive joins of the hexagon $BC'AB'CA'$ are the lines a, b', c, a', b, c'; the lines d, d' are then, respectively, the common transversals of a', b', c', and of a, b, c, and the figure can be completed. The fundamental points are then $F=\frac{1}{2}(B+C-A)$, $R=\frac{1}{2}(B'+C'-A')$, etc. The various relations and notations are represented by the diagram annexed; the fundamental importance of the figure justifies this lengthy description. We shall for clearness denote this figure by Ω. It depends on twenty-four constants. We do not now consider the dually corresponding figure in four dimensions. We consider however the figure obtained by projecting the figure Ω into space of

three dimensions, which we denote by S. It can be constructed, as in the diagram annexed, or as in the diagram given in the Frontispiece, by taking four points which lie in a plane, those denoted by 23, 35, 56, 62; then drawing, through 23, the two arbitrary lines b' and c, and, through 56, the two arbitrary lines b and c'; then, from 35, drawing the line d to meet b' and c', and the line a' to meet b and c, and also, from 62, drawing the line d' to meet b and c, and the line a to meet b' and c'. The remaining incidences of the figure then follow necessarily[1], as the reader may easily see. This figure



depends on nineteen constants; so that there are ∞^1 possible figures when the six fundamental points, from which it can be constructed, are given. The figure in three dimensions which is the dually corresponding to S will be denoted by S'; it is the figure mainly considered by Cremona (*loc. cit.*), containing fifteen lines lying in threes in fifteen planes, of which three pass through each line. Finally we consider the figure obtained by projecting the figure S' on to a plane, which we denote by ϖ; it is in this figure, ϖ, that the various incidences which arise from Pascal's hexagram are found, and for a figure more general than Pascal's. This plane figure consists of two quadrilaterals, say $ABCD$, $A'B'C'D'$, the intersections of corresponding sides being X of AB and $A'B'$, Y of BC and $B'C'$, Z of CD and $C'D'$ and U of DA and $D'A'$, which are such that the points of intersection, O of XZ and YU, P of AC' and $A'C$ and

[1] A proof of this is given, *Proc. Roy. Soc.* LXXXIV, 1911, p. 599.

Q of BD' and $B'D$, are in line. This figure depends on fifteen constants; so that there are ∞^3 possible figures when the six fundamental points, from which it can be constructed, are given.

The correspondence between the figures and the Pascal configuration can now be described precisely, as in the preceding section. If we take a particular order of the systems, say $PQ'RP'QR'$, and consider either the figure Ω, or the figure S, it is clear that the lines $Q'R$, QR', or m and c', are those which join the points 64, 65 to the point 13, the lines RP', $R'P$, or l and c, are those which join the points 65, 64 to the point 23, and the lines $P'Q$, PQ', or a' and b, are those which join the points 64, 65 to the point 12. Thus we have three T-planes passing through the line 64 . 65. In the figure S', dually corresponding to S, we have correspondingly three T-points lying on a line. Projection of S' into the figure ϖ gives then three T-points lying on a line, as in the Pascal configuration. More particularly, each of the systems P, Q', ... consists of five lines, both in the figure S and the figure S'; the systems Q, R' have a definite line in common, as also have the systems Q', R. In order to project S' into ϖ, we must take a definite centre of projection, say O; from this, a transversal can be drawn to these two lines QR' and $Q'R$; it is the intersection of this with the plane of ϖ which gives the T-point. The consecutive lines obtained by the projections of the lines PQ', $Q'R$, RP', $P'Q$, QR', $R'P$ on to the plane of ϖ, similarly give six lines forming a hexagon; the three points of intersection of the pairs of opposite sides of this lie on a line. The vertices of the hexagon in the plane of ϖ therefore lie on a conic. If we take another order of the systems, for instance, $PQ'RP'R'Q$, and project from the same point O, we obtain another hexagon, whose vertices lie on a conic, with its Pascal line. This hexagon will not coincide with the former; in the particular instance, we shall have, instead of the vertex obtained by the intersection of the lines which are the projections of RP' and $P'Q$, the vertex obtained by the projections of the lines RP' and $P'R'$; and so on. As we shew in section (4) below, it is however possible so to specialise the figure as to obtain always the same conic in the plane ϖ.

(3) Dealing now with the figure Ω, or S, we have the fifteen points such as 12, which we may call the Cremona points; we also have the fifteen lines, each containing three of these points whose symbols form a syntheme, such as 12 . 34 . 56. Any two of the Cremona points whose symbols have a number in common may be joined by a line; for instance the line joining the points 12 and 13 is such a line. These lines we call the Pascal lines; their number is sixty. A plane containing three Cremona points whose duad symbols are formed with five of the six numbers, as for instance the plane 23 . 64 . 65, is called a T-plane; it contains four Pascal lines (64 . 65;

46 . 41 ; 56 . 51 ; 14 . 15), and three such planes pass through any Pascal line. In all there are forty-five such planes. There are however also sets of four Pascal lines which meet in a point. For instance the points 64, 61 are those which, in terms of the six fundamental points F, G, H, R, S, T, have the symbols $T+R, T+F$. The line joining these contains the point $R-F$; this we denote by [14]. Evidently there are fifteen such points, which we call Plücker points; and, for instance, through the point [14], there pass the four Pascal lines 21 . 24, 31 . 34, 51 . 54 and 61 . 64. The Plücker points lie in threes upon twenty lines; for instance the points [23], [31], [12], whose symbols are, respectively, $G-H, H-F, F-G$, lie upon a line. Such a line is called a *g*-line, or a Cayley-Salmon line. The Plücker points also lie in sixes upon fifteen planes. For instance the points [63], [64], [65], [45], [53], [34] lie on a plane. Such a plane is called an *I*-plane, or a Salmon plane. The Pascal lines lie in threes in planes, in two distinct ways. A plane containing three Cremona points whose duad symbols contain only three of the numbers, for instance the plane of the points 23, 31, 12, evidently contains the three Pascal lines 12 . 13, 23 . 21, 31 . 32. Such a plane is called a *G*-plane, or a Steiner plane; the total number of such planes is twenty. Again a plane containing three Cremona points whose duad symbols all contain one number in common, for instance the plane of the points 41, 42, 43, evidently contains the three Pascal lines 42 . 43, 43 . 41, 41 . 42. Such a plane is called a *K*-plane, or a Kirkman plane; the total number of such planes is sixty. Lastly, it is necessary to refer to fifteen lines, each joining a Cremona point to the Plücker point whose symbol is formed with the same numbers, for instance the line joining 12 to [12]. Such a line is called an *i*-line, or a Steiner-Plücker line. The Pascal lines will also be called *k*-lines. There are certain theorems of incidence among the various elements in addition to those which have been referred to. For instance, a Pascal line, or *k*-line, lies in one *G*-plane, and in three *K*-planes; this fact will be denoted, in the summary we now give, by writing *k*...*G*, 3*K*. Conversely a *K*-plane contains, not only three *k*-lines, as we have seen, but also one *g*-line; this fact will be denoted by writing *K*...*g*, 3*k*. We may then summarise the definitions, and the incidences referred to, which will be immediately proved, as follows: [Add.]

60 *k*-lines, Pascal lines, 12, 13,
60 *K*-planes, Kirkman planes, 14, 15, 16;

20 *g*-lines, Cayley-Salmon lines, [12], [13],
20 *G*-planes, Steiner planes, 56, 64, 45;

15 *i*-lines, Steiner-Plücker lines, 12, [12],
15 *I*-planes, Salmon planes, [63], [64], [65].

$k \ldots G, 3K$; $g \ldots G, 3K, 3I$; $i \ldots 4G$,
$K \ldots g, 3k$; $G \ldots g, 3k, 3i$; $I \ldots 4g$.

The proof of these last theorems of incidence is immediate:

(1) The k-line 12, 13 lies in the G-plane 23, 31, 12; and in the K-planes (12, 13, 14), (12, 13, 15), (12, 13, 16).

(2) The K-plane 14, 15, 16 contains the g-line ([56], [64], [45]), and the k-lines 15 . 16, 16 . 14, 14 . 15.

(3) The g-line [12], [13] lies in the G-plane (23, 31, 12), it lies in the K-planes (41, 42, 43), (51, 52, 53), (61, 62, 63), and it lies in the I-planes ([12], [13], [14]), ([12], [13], [15]), ([12], [13], [16]).

(4) The G-plane 56, 64, 45 contains the g-line [56], [64], [45], the k-lines 45 . 46, 56 . 54; 64 . 65, and also the i-lines (56, [56]), (64, [64]), (45, [45]).

(5) The i-line (12, [12]) lies in the G-planes (12, 23, 31), (12, 24, 41), (12, 25, 51), (12, 26, 61).

(6) The I-plane ([63], [64], [65]) contains the g-lines ([63], [64], [34]), ([64], [65], [45]), ([65], [63], [35]), ([45], [53], [34]).

It is interesting, however, further, to remark, that the relations expressed by

$K \ldots g, 3k$; $G \ldots g, 3k, 3i$; $I \ldots 4g$

can be verified by considering only a single threefold space which forms part of Ω, there being in Ω fifteen such spaces; and the relations expressed by

$k \ldots G, 3K$; $g \ldots G, 3K, 3I$; $i \ldots 4G$

can be verified by considering only elements passing through a single point of Ω, there being fifteen such points, which correspond dually in Ω to the threefold spaces just spoken of. For if we consider the space determined by four of the six fundamental points, say by G, H, S, T, which contains the points 23, 25, 26, 35, 36, 56, [23], [25], [26], [35], [36], [56], it is at once seen that this contains

four K-planes, four G-planes, one I-plane;

it contains also

twelve k-lines, four g-lines, six i-lines.

Correspondingly, consider elements passing through the Plücker point [14]; there are of such (cf. the Frontispiece of the volume)

four k-lines, four g-lines, one i-line;

any plane of the complete figure Ω which contains one of these lines necessarily contains the point [14]. There pass in all through this point

twelve K-planes, four G-planes, six I-planes;

one such K-plane is (21, 24, 23); one such G-plane is (21, 24, 14); one such I-plane is ([14], [15], [16]).

And in fact, if we introduce the equations to the spaces and planes of the figure Ω, we can make the duality suggested by the notation apparent analytically.

If now we suppose that in place of the figure S we consider the figure S', dually corresponding to S in three dimensions, we shall have therein k-lines, g-lines and i-lines, but we shall have *K-points*, instead of K-planes, G-points and I-points. If then we project on to a plane, obtaining the figure ϖ, we shall have therein lines from the lines of the figure S', and points from the points of S'. And therein the six theorems of incidence which we have obtained will continue to hold.

It is this fact which is the outcome of the many investigations made for Pascal's figure, of which the figure ϖ is a generalisation.

(4) We shew now how the exact Pascal's figure can be obtained from a figure in three dimensions; and it will appear at once that this figure is a particular case of the figure S' which we have considered.

Let x, y, z, t be coordinates in three dimensions. Let U be a homogeneous polynomial of the second order in x, y, z only, and M be a homogeneous polynomial of the third order in x, y, z only. Thus $U=0$ may be regarded as the equation to a conic, in the plane whose equation is $t=0$; similarly $M=0$ may be regarded as the equation to a curve in this plane which has the property of being met, by an arbitrary line of this plane, in three points, and is therefore said to be a curve of the third order, or a cubic. This cubic meets the conic in six points, as follows from the elements of the theory of elimination. These points we name, in some order, P, Q', R, P', Q, R'; a cubic curve can be drawn through nine arbitrary points, so that, by proper choice of M, we may regard these six points as arbitrary points of the conic $U=0$. Now consider the equation $M-tU=0$, which is homogeneous in the four coordinates x, y, z, t. It represents a surface which is met by an arbitrary line of the threefold space in three points, say, a cubic surface. But, in particular, any line of the threefold space which is drawn through the point of coordinates (0, 0, 0, 1), say, the point D, meets the surface in two coincident points at D, and in a further point; for this reason the point D is called a node of the cubic surface. More particularly, however, there are lines which lie entirely upon the cubic surface: for instance, whatever θ may be, if $(x, y, z, 0)$ be one of the six points for which $U=0$ and $M=0$, the point (x, y, z, θ) lies on the cubic surface, since its coordinates satisfy $M-\theta U=0$, and, for different values of θ, this point is any point of the line joining D to $(x, y, z, 0)$. Let the six lines so obtained, joining D to the six points $P, Q', \ldots$, be named, respectively, p, q', r, p', q, r'; it is easy to see that these are the only lines passing through D which lie on the cubic surface. Then, further, it is at once clear that any plane meets the cubic surface in a cubic curve; thus a plane containing two of the lines $p, q', \ldots, r'$, will contain a further line lying entirely on the cubic surface. Thereby we obtain fifteen further lines of the surface, any one of which may be denoted by a duad symbol, such as qr'. There are however no other lines lying on the surface. For we have enumerated all those which pass through D; suppose then one which does not pass through D, and consider the curve in which the cubic surface is met by the plane joining D to this line; this consists of the line, and a curve of the second order; but this curve of the second order has the property of being met in two points coinciding at D by every line in the plane drawn through D, and must therefore itself consist of two lines intersecting at D. The plane is therefore one of those before considered, drawn through two of the lines $p, q', \ldots, r'$. Now consider the three lines of the surface, q, r' and qr', which lie in a plane. The first line, q, is evidently met by the four lines qp, qr, qp', qq', in addition to qr'; the other line, r', is evidently met by the four lines $r'p$, $r'r$, $r'p'$, $r'q'$, in addition to $r'q$. Beside the four lines which meet both q and r' at the point D, there remain then, of the total twenty-one lines of the surface, just six, namely those whose symbols are the duads from p, r, p', q'. These six lines do not lie in the plane of q, r' and qr'; each must meet, then, either q, or r', or qr'; but each meets two lines, other than q and r', of the lines of the surface which pass through D; it cannot, then, for example, meet also the line q, since else three of the lines through D would lie in a plane, and the conic $U = 0$ would break into two lines, which we suppose not to be the case. Thus these six lines all meet the line qr'.

Consider, for instance, rp'; the plane containing qr' and rp' must then meet the cubic surface in another line, by an argument applied above; this other line, meeting both qr' and rp', will have for symbol a duad not containing either q or r', or r or p'; this symbol is, then, pq', the symbols of the three lines qr', rp', pq' containing all the six symbols p, q', r, p', q, r'. In this way we see that there are, beside the plane containing q and r', three planes through the line qr' each containing two other lines with duad symbols. Whence we see that the fifteen lines of the cubic surface, other than those through D, lie in threes in fifteen planes, of which three planes pass through every one of these lines.

If desired, the equations of the fifteen lines can be obtained, when the six points of intersection of the conic $U = 0$, and the cubic $M = 0$ are given. For if $ax + by + cz = 0$, with $t = 0$, be the equation of the join of two of these six points, and we suppose, for instance, that c is not zero, as the plane joining D to this line contains a line of the cubic surface, the substitution for z, in the quotient M/U, of $-(ax+by)/c$, must reduce this to a form $lx+my$; the line in question will then be given by the two equations $ax+by+cz=0$, $t-lx-my=0$.

Having obtained the figure of fifteen planes, and the lines lying in threes of these, we can denote any one of the planes by a duad formed from two of the six numbers 1, 2, 3, 4, 5, 6, and the line of intersection of three of these planes by a syntheme of three duads, such as 12 . 34 . 56, formed from the duads which represent the three planes passing through the line. We shall then have a figure which, combinatorially, has the same properties as the figure S' considered above, specialised however by the fact that the fifteen lines are arranged in sets of five all of which have a common transversal passing through D, any of the lines meeting two of the transversals. These sets of five are the systems of synthemes considered above in (1). But the incidence theorems obtained above will hold for the specialised figure; and, after projection on to the plane $t=0$, these give the properties of the Pascal figure which are the occasion of the present note.

It may be remarked that on a general cubic surface, which does not possess a node, there are sets of fifteen lines such as those here denoted by the duad symbols qr', together with twelve others, each of the six lines here found passing through D being replaced by a pair of skew lines. Moreover, we shall find that the general figure Ω in four dimensions, here considered, is of fundamental importance for the theory of the general cubic surface.

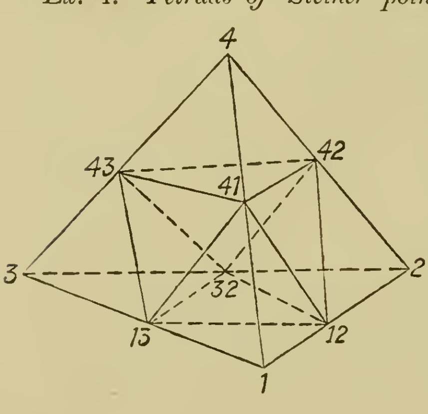
(5) We now give some particular examples of the theory.

Ex. 1. *The pairs of conjugate Steiner planes.* Let $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ denote the numbers 1, 2, ..., 6, in any order. In the figure Ω, there pass, as we have seen, through each of the three Cremona points, $\beta'\gamma'$, $\gamma'\alpha'$, $\alpha'\beta'$, three of the fifteen lines which are fundamental in the figure (beside two k-lines joining this point to the other two); and the nine lines so obtained meet in threes in the points $\beta\gamma, \gamma\alpha, \alpha\beta$. The two Steiner planes $(\beta'\gamma', \gamma'\alpha', \alpha'\beta')$ and $(\beta\gamma, \gamma\alpha, \alpha\beta)$ are called conjugate. In the Pascal figure there are two corresponding Steiner points, which are in fact conjugate to one another in regard to the fundamental conic. In the figure S', we have two Steiner points; through each of these there pass three Cremona planes, intersecting in pairs in three Pascal lines which pass through the point. The three planes through one of the two Steiner points meet the three planes through the other Steiner point in three triads of lines, of which any two are then in perspective, the axis of perspective being a Pascal line. By projection to the plane ϖ we obtain then a similar result, of which the elementary proof is indicated in more detail below. (Ex. 5. See also v. Staudt, *Crelle*, LXII, p. 142.)

Ex. 2. *The Steiner planes and Steiner-Plücker lines.* The plane determined by the Cremona points 23, 31, 12, is the same as that determined by the three points F, G, H, of the six fundamental points. The Steiner-Plücker line determined by joining the Cremona point 12 to the Plücker point [12], is that joining the two fundamental points F, G. The twenty Steiner planes are thus all the planes each containing three of the six fundamental points, and the fifteen Steiner-Plücker lines are all the lines each joining two of these points. In the figure S', the Steiner points are then the intersections in threes, and the Steiner-Plücker lines the intersections in twos, of six planes, which, so far as the configuration is concerned, may be taken arbitrarily. The corresponding points and lines in the Pascal figure are then the projection of this very simple configuration. (Cf. p. 218.)

Ex. 3. *The separation of the complete figure into six figures.* In the figure Ω, the notation at once suggests that we consider together the set of five Cremona points such as 12, 13, 14, 15, 16; and there will be six such sets, each point belonging to two sets. The five points of a set are such that the join of any two is a Pascal line, and the plane of any three is a Kirkman plane. In the figure S', we have then six sets of five Cremona planes, of which any two meet in a Pascal line, and any three in a Kirkman point. In the plane ϖ, the sixty Pascal lines are thus divided into six sets of ten, meeting in threes in ten Kirkman points, each of the Pascal lines containing three of the Kirkman points. The configuration, in each of the six partial figures is that of two triads in perspective, with their centre and axis of perspective. These partial figures were considered by Veronese.

Ex. 4. *Tetrads of Steiner points each in threefold perspective with a tetrad of Kirkman points.*



In the figure Ω, the six Cremona points 23, 31, 12, 41, 42, 43, which lie in the threefold space of the four fundamental points F, G, H, R, give four Steiner planes,

$\alpha=(24,43,32)$, $\beta=(34,41,13)$, $\gamma=(14,42,21)$, $\delta=(23,31,12)$, and four Kirkman planes,

$\alpha'=(12,13,14)$, $\beta'=(23,21,24)$, $\gamma'=(31,32,34)$, $\delta'=(41,42,43)$.

The planes α, α' meet in the

g-line ([23], [24], [34]), the planes β, β' meet in the g-line ([31], [34], [14]), the planes γ, γ' meet in the g-line ([12], [14], [24]), and the planes δ, δ' meet in the g-line ([12], [13], [23]); and these are the four g-lines which lie in the I-plane ([41], [42], [43]).

Hence if we consider the figure S', we have two tetrads of points, one formed by Steiner points, and the other by Kirkman points, which are in perspective, the lines joining corresponding points of the two tetrads being g-lines, and the centre of perspective being an I-point.

But in fact, in this figure S', the two tetrads are in perspective in three other ways. In this figure, α, β, ..., α', β', ... are points; the lines (α, δ'), (β, γ'), (γ, β'), (δ, α') also meet in a point, namely the point (12, 13, [14]), or (12, 13, 56), a T-point. This fact we may denote by writing

$$12,\ 13,\ [14] \begin{cases} 24, 43, 32 \quad 34, 14, 31 \quad 14, 24, 12 \quad 23, 31, 12 \\ 41, 42, 43 \quad 31, 32, 34 \quad 23, 21, 24 \quad 12, 13, 14 \end{cases}$$

The joins of corresponding points of these two tetrads are, respectively, the Pascal lines 42 . 43, 31 . 34, 21 . 24, 12 . 13.

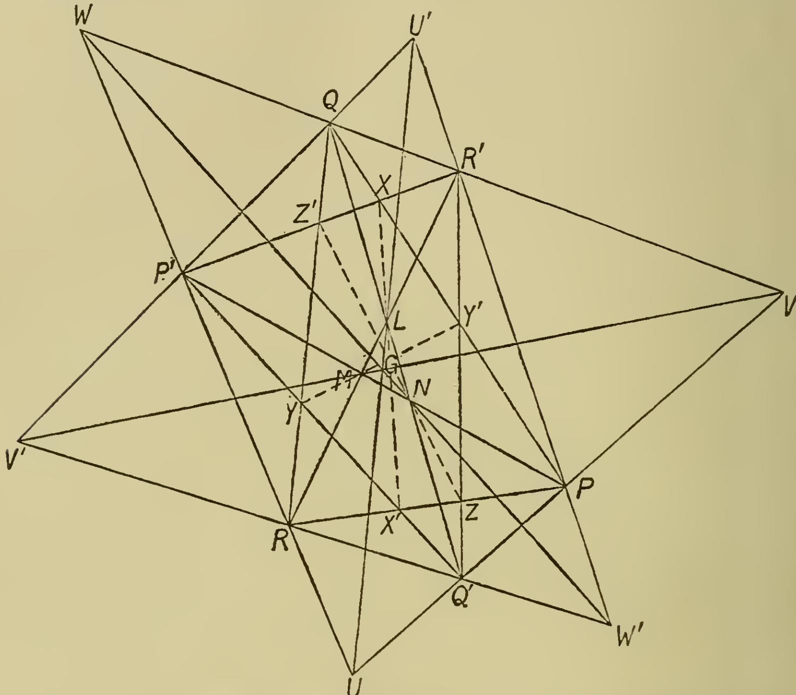
Similarly the tetrads $(\alpha, \beta, \gamma, \delta)$, $(\gamma', \delta', \alpha', \beta')$ are in perspective from the point (23, 21, [24]), or (23, 21, 56), and the tetrads $(\alpha, \beta, \gamma, \delta)$, $(\beta', \alpha', \delta', \gamma')$ are in perspective from (31, 32, [34]), or (31, 32, 56). The two given tetrads, and that formed by the four centres of perspective, form, therefore, what is known as a desmic system of three tetrads, of which the points can be supposed to have symbols of the form (cf. p. 213)

$$(P,Q,R,S),(-P+Q+R+S,P-Q+R+S,P+Q-R+S,P+Q+R-S),$$
$$(P+Q+R+S,\ P-Q-R+S,\ -P+Q-R+S,\ -P-Q+R+S).$$

Ex. 5. *Relation of the Cayley-Salmon lines and the Salmon planes.* The fifteen Plücker points in the figure Ω lie in threes on the twenty Cayley-Salmon lines, g-lines, and in threes also on the fifteen Salmon planes, or I-planes, forming a figure such as that occurring in the proof of Desargues' theorem for two triads in one plane (cf. p. 214).

The figure is evidently dual with itself, in threefold space, and has the same description in S' and in ϖ. Through each of the fifteen points there pass, in S or S', six planes and four lines, each of the twenty lines contains three points and lies in three planes, each plane contains four of the lines and six of the points. The figure has been considered in Note I (§ 3).

Ex. 6. Let U, V, W and U', V', W' be two triads of points in a plane, the lines UU', VV', WW' meeting in G; let the points $(UV, W'U')$, $(VW, U'V')$, $(WU, V'W')$ be denoted, respectively, by P, Q, R, and the points $(U'V', WU)$, $(V'W', UV)$, $(W'U', VW)$ be denoted, respectively, by P', Q', R'; so that, by the converse of Pascal's theorem, the six points P, Q', R, P', Q, R' lie on a conic. Prove, (1), that QQ', RR', UU' meet in a point, say, L; and, similarly, RR', PP', VV' meet in a point, say, M; and, likewise, PP', QQ', WW' meet in a point, say, N. Let the points $(PQ, R'P')$, $(QR, P'Q')$, $(RP, Q'R')$ be denoted, respectively, by X, Y, Z; and the points $(P'Q', RP)$, $(Q'R', PQ)$, $(R'P', QR)$ be denoted, respectively, by X', Y', Z'. Prove, (2), that X, L, X' are in line, as also Y, M, Y' and Z, N, Z'; and that



the lines XLX', YMY', ZNZ' meet in a point. Prove also, (3), that the lines YZ, PP', QR' meet in a point, as do the lines $Y'Z', PP', Q'R$; that the lines ZX, QQ', RP meet in a point, as do $Z'X', QQ', R'P$; and that the lines XY, RR', PQ' meet in a point, as do the lines $X'Y', RR', P'Q$. Thus any two of the triads U, V, W; L, M, N; X, Y, Z are in perspective, as also are any two of the triads U', V', W'; L, M, N; X', Y', Z', the lines XU, YV, ZW meeting in a point, as do $X'U', Y'V', Z'W'$.

Shew also, (4), that the lines UU', VV', WW' are Pascal lines each for a proper order of the six points P, Q', etc., on the conic, their point of meeting, G, being a Steiner point; and, (5), that the axis of perspective of the triads U, V, W and U', V', W' is a Pascal line;

that the axis of perspective of the triads U, V, W and L, M, N is a Pascal line, and that the axis of perspective of the triads U', V', W' and L, M, N is a Pascal line; and that these three Pascal lines meet in a Steiner point. Shew also, (6), that the lines XLX', YMY', ZNZ' are Pascal lines, meeting in a Kirkman point, that the lines XU, YV, ZW are Pascal lines, meeting in a Kirkman point, and that the lines $X'U'$, $Y'V'$, $Z'W'$ are Pascal lines, meeting in a Kirkman point.

These facts may all be obtained by application of Desargues' theorem.

They may also be obtained by shewing that, with that association of P, Q', R, etc. with the systems which has been adopted in this note, the points U, V, W, U', V', W' are the respective T-points (56, 21, 23), (56, 12, 13), (56, 31, 32), (64, 21, 23), (64, 12, 13), (64,31,32), so that the lines UU', VV', WW' are the respective Pascal lines (23,21), (12,13), (31, 32). Then that the points L, M, N are the respective T-points (45,23,21), (45,12,13), (45,13,23). Then that the points X, Y, Z are the respective T-points (25, 63, 61), (15, 62, 63), (35, 61, 62), and the points X', Y', Z' are the respective T-points (24, 61, 63), (14, 62, 63), (34, 61, 62), so that the lines XX', YY', ZZ' are, respectively, the Pascal lines (63, 61), (62, 63), (61, 62) and the lines YZ, $Y'Z'$ are, respectively, the Pascal lines (24, 26), (25, 26), with similar results for ZX, $Z'X'$, XY, $X'Y'$. The points (QR', PP'), (RP', QQ'), (PQ', RR') are, respectively, the T-points (31, 56, 54), (23, 56, 54), (12, 56, 54); the lines XU, $X'U'$, YV are, respectively, the Pascal lines (41, 43), (51, 53), (42, 43), and so on.

(6) For the literature of the matter, the reader may consult:

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L. Wedekind, *Math. Ann.* XVI (1880), p. 236, with a bibliography.

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