The figure formed from six points in space of four dimensions.

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It was my aim when commencing this sketch to illustrate the fact that a very large proportion of the known properties of certain much studied families of points, both in a plane and in space, (and therefore also of the reciprocal families of lines and planes), are intuitive consequences of the nature of a very simple figure in space of four dimensions, — the figure derived by the simplest operations of geometry from six points chosen at random in such space. The families of lines and points and planes referred to include (1) the fifteen lines which join six points of a curve of the second degree, — Pascal's Hexagram; (2) certain sets of fifteen of the double tangents of a plane curve of the fourth order; (3) fifteen points in space which are nodes of a surface of the fourth order; (4) any fifteen of the nodes or of the singular planes of Kummer's quartic surface; as well as other families less known than these. To extend or even to quote the long list of published properties of these figures was no part of my plan; for the tendency of recent times is, very rightly, to turn in weariness from the apparently interminable successions of elementary theorems which geometry sometimes presents to us. But, on the other hand, to prove that practically all results hitherto established for the simplest and most fully investigated of the above families belong in a wider sense to all, and to refer these results to a common simple cause, appeared to me a legitimate theme. In concluding this note I find myself rather of a mind to enlarge upon the fundamental necessity for the existence of families of points possessed of these properties, as an immediate consequence of the axioms of four-dimensional space, that four points determine a single space of three dimensions and so forth: yet so large a part of this sketch consists of a review of the work of others, — as references given in the text will shew, — that the somewhat illogical method pursued was almost forced upon me. From results already established concerning Pascal's Hexagram, cubic surfaces, and quartic curves, a system of equations is derived, which are seen to apply properly to space of four dimensions, and are so interpreted: the final conclusion reached by this method is that the properties so obtained are fundamental, and might be considered, without reference to coordinates or curved loci, at the outset of descriptive geometry.

§ 1.

On Pascal's Hexagram.

Although Pascal's original theorem — that, when the vertices of a hexagon lie on a curve of the second degree, the intersections of its opposite sides are collinear, — dates from as long ago as 1640, no advance towards the development of the figure now commonly known as Pascal's Hexagram was made till the present century. Brianchon's theorem concerning six tangents of a curve of the second class, obtained by reciprocation in 1806, cannot, according to modern ideas, be ranked as distinct from that of Pascal; the field for nearly all later research was opened by Steiner in 1828, when he pointed out that from the same six points of a conic sixty hexagons may be formed, each of which gives rise to a different Pascal line. During the next fifty years the figure of these sixty lines attracted the attention of Steiner, Plücker, Cayley, Salmon, Kirkman, Hesse, v. Staudt, Grossman, Bauer, Schröter, and others, until finally in 1877 the results of their investigations were summed up and extended by Veronese, (*Atti della R. Accad. dei Lincei*, vol. 1, series III, pp. 649—703). Inasmuch as Veronese's memoir, (which is prefaced by an excellent historical sketch of the subject, with full references to the works of earlier writers), contains proofs of all previously known theorems as well as of a large number of new and original ones, we shall class together all these properties under the name *Veronese's properties of Pascal's Hexagram.*

The same volume of the *Atti della R. Accad. dei Lincei* contains a second memoir (pp. 854—874) of even greater importance, which throws an entirely new light upon the nature of the figure. Cremona, to whom Veronese had submitted his manuscript, was led on reading it to the discovery that the whole series of theorems established therein follows intuitively from the obvious properties of the lines which lie upon a surface of the third order with a nodal point. On such a surface lie six lines passing through the nodal point, generators of the quadric cone which touches the surface there, and fifteen others, one in the plane of each pair of the foregoing: let the former be denoted by symbols a, b,

c, d, e, f, and the latter by pairs of these symbols $ab, ac, \ldots ef$, in such a way that ab is the line which meets the lines a and b of the former set. If now this system of lines be projected upon a plane, the nodal point being the vertex of projection, the plane of the projected system cuts the lines a, b, c, d, e, f in six points (also called a, b, c, d, e, f) situated on a curve of the second degree; and the projection of the line ab is a line (also called ab) which joins the points a and b: we thus have fifteen lines which join in pairs six points of a conic section, the foundation from which the complete figure of Veronese's memoir is built up.

But the plane figure is in reality far less simple than the three-dimensional. In the former the fact that every two lines intersect causes unnecessary confusion; for it appears upon examination that Veronese's memoir nowhere contains any property concerning the point of intersection of two lines, except when the lines of which they are the projections actually intersect; not only this, but the nature of the three-dimensional figure renders obvious all (or very nearly all) Veronese's results: — thus for example a proposition that three lines are concurrent needs no further proof when it can be pointed out that they are projections of the lines of intersection of three planes: — Cremona's three-dimensional figure in fact contains all that is essential to the proof of Veronese's theorems and is free from what is irrelevant; moreover the vast numbers of lines and points which make up the plane figure are obtained by projecting the intersections of a comparatively small number of planes in space. Cremona's methods are purely geometrical, but the investigation is very easily conducted by help of an extremely simple and symmetrical system of equations; — not those given by Cremona at the end of the memoir to which we have already referred, but a system derivable immediately from the form to which he has elsewhere reduced a non-singular surface of the third order: see *Math. Annalen* XIII, p. 301; or *Salmon-Fiedler, Anal. Geom. des Raumes*, p. 403: a full investigation of Veronese's results will be found in the *Transactions of the Cambridge Philosophical Society*, Vol. XV, p. 207 from which I now quote the following.

Given a surface of the third order, having a nodal point and no further singularity, a unique family of six planes $x_1 = 0, x_2 = 0, \ldots \ldots x_6 = 0$, may be determined, such that the surface is represented by the equation

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 = 0;$$

or

$$\Sigma(x_r^3) = 0; \quad (r = 1, 2, 3, 4, 5, 6);$$

with the following conditions,

$$\Sigma(x_r)\equiv 0;\quad \Sigma(k_r^2 x_r)\equiv 0;\quad \Sigma(k_r)=0;\quad \Sigma(k_r^3)=0:$$

the constants $k_1, k_2, \ldots k_6$, are the coordinates of the conical point, and the two identical linear relations which connect the six coordinates x_r of any point are stated explicitly. The six lines a, b, c, d, e, f, which pass through the nodal point are the intersections of the surface with the tangent cone, $\Sigma(k_r x_r^2)=0$, and need not be further particularized; while each of the fifteen lines, $ab, ac, \ldots ef$, is represented by one of a family of fifteen exactly similar equations of which

$$x_1+x_2=x_3+x_4=x_5+x_6=0$$

is the type: the lines ab, ac, etc. therefore lie by threes in the fifteen planes $x_1+x_2=0$, according to the following or some equivalent scheme: —

ab, cd, ef in $x_1+x_2=0$	af, bd, ce in $x_2+x_3=0$	ad, bc, ef in $x_3+x_5=0$
$ac, be, df \ldots x_1+x_3=0$	$ae, bc, df \ldots x_2+x_4=0$	$ae, bf, cd \ldots x_3+x_6=0$
$ad, bf, ce \ldots x_1+x_4=0$	$ac, bf, de \ldots x_2+x_5=0$	$af, be, cd \ldots x_4+x_5=0$
$ae, bd, cf \ldots x_1+x_5=0$	$ad, be, cf \ldots x_2+x_6=0$	$ac, bd, ef \ldots x_4+x_6=0$
$af, bc, de \ldots x_1+x_6=0$	$ab, cf, de \ldots x_3+x_4=0$	$ab, ce, df \ldots x_5+x_6=0$

all other possible schemes can be derived from this by suitable interchange either of the symbols a, b, c, d, e, f, or of the suffixes 1, 2, 3, 4, 5, 6; the equations of any one of the fifteen lines can be at once selected. A few of the chief properties of the Hexagram are proved below, the names of the various types of lines and points being in all cases adopted from Veronese's memoir.

(α) If a hexagon be constructed whose vertices are the six points a, b, c, d, e, f, (which lie on a conic) taken in any order, the opposite sides intersect in three points situated in one of sixty Pascal lines. For example, in the case of the hexagon $abefdc$, the three alternate sides ab, ef, dc are projections of lines which lie in the plane $x_1+x_2=0$, and be, fd, ca are projections of lines which lie in $x_1+x_3=0$; therefore the intersections of ab with fd, of be with dc, and of ef with ca, being projections of three points which lie on $-x_1=x_2=x_3$, the line of intersection of the two planes, are themselves collinear. The symmetry of our system of equations enables us to infer, by interchanging the suffixes 1, 2, 3, 4, 5, 6, in all possible ways, that there are sixty of these Pascal lines, each the projection of a line such as $-x_1=x_2=x_3$.

(β) The Pascal lines meet by threes in sixty Kirkman points, there being a correspondence between each Kirkman point and one particular Pascal line; (the projection of the point $-x_1=x_4=x_5=x_6$ is a Kirkman point, to which the Pascal line given in (α) corresponds).

When three Pascal lines meet in a Kirkman point, their corresponding Kirkman points lie on a Pascal line. These points and lines fall into six figures of ten Pascal lines and ten Kirkman points: each figure consists of the projections of the intersections of five planes such as

$$x_1 + x_2 = 0, \quad x_1 + x_3 = 0, \quad x_1 + x_4 = 0, \quad x_1 + x_5 = 0, \quad x_1 + x_6 = 0,$$

and may be resolved in ten different ways into two perspective triangles with their axis of homology.

(γ) The Pascal lines meet also by threes in twenty Steiner points, which lie by fours on fifteen Steiner-Plücker lines: these are projections of points such as $x_1 = x_2 = x_3 = 0$, and of lines such as $x_1 = x_2 = 0$, respectively, the edges and vertices of the figure formed by the six planes $x_1 = 0,\ x_2 = 0, \ldots x_6 = 0$: the figure formed by the Steiner points and Steiner-Plücker lines may therefore be resolved into three perspective triangles with their three concurrent axes of homology in twenty different ways.

(δ) When three Pascal lines meet in a Steiner point, the corresponding Kirkman points lie on one of twenty Cayley-Salmon lines which meet by fours in fifteen Salmon points: projections of $x_4 = x_5 = x_6$, and of $x_3 = x_4 = x_5 = x_6$ respectively.

(ε) Six lines such as bc, ca, ab, ef, fd, de, touch a conic: for they are projections of six lines of which each of the first three meets each of the last three, and which are therefore generators of a quadric surface: in fact $x_1^2 + x_4^2 + x_5^2 = x_2^2 + x_3^2 + x_6^2$.

The system of equations here used, being unique and absolutely symmetrical in form, gives an immediate answer to all questions of correspondence between the different lines and points, or the number of those of a given type, and all Veronese's results may be verified without difficulty. Upon closer investigation I find that by projecting the complete system of intersections of the two families of planes $x_1 \pm x_2 = 0$, — called by Cremona 'tritangent' planes and Plücker planes respectively, — a figure is obtained which in all except a few unimportant details is coextensive with that developed by Veronese: together with proofs, similar to those just quoted, of practically all his theorems. There is a certain slight advantage gained by applying the name Pascal's Hexagram to the figure so defined: we may do so without injustice to earlier writers: while it seems desirable that the statement which will repeatedly be made in the next section, that a family of lines possesses the properties of Pascal's Hexagram, should carry with it a clear and definite meaning: it is not implied that the list of results shared by these families of lines and the Hexagram cannot be extended.

§ 2.

On plane systems of lines which posses Veronese's properties of the Hexagram.

It will be observed that, in the course of the foregoing verification of Veronese's results, no use whatever is made of the conditions $\Sigma(k_r^2 x_r) \equiv 0$ and $\Sigma(k_r^3) = 0$, which connect the coordinates (x) and the constants (k): it is immaterial whether these conditions are satisfied or not. In order that a family of coplanar lines should possess all the properties established in Veronese's memoir for the fifteen lines which join six points of a conic, all that is necessary is that they should be projections of fifteen lines in space that satisfy the family of equations similar to

$$(1) \qquad \left\{ \begin{array}{l} x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = 0, \\ \text{when} \\ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \equiv 0 \end{array} \right.$$

or, as Cremona expresses it, fifteen lines which lie by threes in fifteen planes. Now it is clear that the family of lines (1) lie upon

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 = 0,$$

a cubic surface without singularities (see Cremona, *Math. Annalen* XIII, p. 301; Salmon-Fiedler p. 403) and form such a set as is left when from the twenty-seven lines of the surface twelve are omitted which form a *double-six*. (Schläfli, *Quarterly Journal of Mathematics*, Vol. 2, pp. 55 and 110.) In his memoir Cremona points out that the omission of a double-six from the lines of a cubic surface would furnish an example of such a family of lines, but goes no further, being apparently not aware that some years earlier Geiser had discussed the projections of the lines of a cubic surface in the well known paper, published in Vol. I of *Math. Annalen* p. 129, in which the connexion by this geometrical method between the lines of a surface of the third order and the double tangents of a plane curve of the fourth order was first made known. If a point K be taken, the cone with vertex K which envelopes the cubic surface is touched at two distinct points by every line which lies on the surface, viz. the two points where the line cuts the first polar of K: every plane section of the enveloping cone will therefore have the projections from vertex K of the twenty seven lines of the surface as double tangents. When the position of K is unrestricted, a plane section of the enveloping cone is a curve of the sixth order having six cusps which lie on a conic: to such a curve belong twenty-seven double tangents, from which we may in thirty-six different ways select a set of fifteen possessing all Veronese's properties of Pascal's Hexagram. But the special case when K lies upon the cubic surface is of greater interest: the section of the cone with vertex K which envelopes the surface is then a curve of the fourth order without singularities; twenty-seven of whose double tangents are projections from vertex K of the lines which lie on the cubic surface, the remaining double tangent being the intersection of the tangent plane at K with the plane of the curve: moreover I find that if Schläfli's notation be adopted for the lines of the cubic surface, and that of Hesse for the double tangents of the quartic curve, (see Crelle, Vol. 49, p. 243 and Vol. 68, p. 176; or Salmon, *Higher Plane Curves*), it is legitimate to denote the projections of Schläfli's $(a_1, a_2, a_3, a_4, a_5, a_6)$, $(b_1, b_2, b_3, b_4, b_5, b_6)$, $(c_{12}, c_{13}, c_{14}, c_{15}, c_{16})$, $(c_{23}, c_{24}, c_{25}, c_{26})$, $(c_{34}, c_{35}, c_{36}, c_{45}, c_{46}, c_{56})$, by Hesse's (ag, bg, cg, dg, eg, fg), (ah, bh, ch, dh, eh, fh), (ab, ac, ad, ae, af), (bc, bd, be, bf), (cd, ce, cf, de, df, ef) respectively, the last double tangent being represented by gh. Omitting from the lines of the cubic the double-six composed of the first twelve of these lines, we arrive at a theorem which may be stated in the following somewhat remarkable form:

The fifteen double tangents of a plane curve of the fourth order denoted in Hesse's Algorithm by symbols formed of pairs of the six letters a, b, c, d, e, f, possess all the properties of the Pascal's Hexagram formed by the lines (naturally represented by the same symbols) which join in pairs six points, a, b, c, d, e, f, of a conic section.

For example, in the case of the double tangents of a quartic curve, just as in the Hexagram, the intersections of ab and fd, of be and dc, of cf and ca lie on a line, one of a set of sixty similar Pascal lines, which meet by threes in sixty Kirkman points and with them fall into six figures of ten Pascal lines and ten Kirkman points, each figure being that familiar to us as the projection of the lines and points of intersection of five planes in space: the Pascal lines also meet by three in twenty Steiner points (projections of the vertices of a three-dimensional figure formed by six planes) which lie by fours in fifteen lines, (projections of its edges):... the double tangents bc, ca, ab, ef, fd, de, touch a conic section;... and so on through the whole category of Veronese's propositions. [The names Pascal line, Kirkman point, Steiner point are adopted from the Hexagram for convenience: also the notation of *Salmon's Higher Plane Curves* is slightly altered, the symbols 1, 2, 3, 4, 5, 6, 7, 8 being there used in place of our a, b, c, d, e, f, g, h.]

A very remarkable fact, not however without parallel, comes to light when we seek the distinctive geometrical properties of such a set of fifteen double tangents. The rule of the *bifid* substitution, (due to Cayley and Hesse, and explained in Salmon), makes it clear that if we select any two of the twenty-eight double tangents, (e. g. ag and ah), and the five pairs (bg, bh; cg, ch; dg, dh; eg, eh; fg, fh) whose points of contact lie on a conic with those of the first two, *any* fifteen of the remaining sixteen form such a set. Thus it appears that, whereas in Pascal's Hexagram, or in the case of double tangents of the sextic curve above mentioned, we have to deal with sets of fifteen lines which possess a long series of properties owing to a quite definite cause, we here find the fifteen lines joined by a sixteenth which forms with them an absolutely symmetrical family, any fifteen of whose members possess all the properties of the former sets. Sense of symmetry alone shews the necessity of considering sets of sixteen double tangents of the quartic rather than fifteen; but it will also be seen that, in asserting that the fifteen double tangents ab, ac, ... ef, possess all the properties of the Hexagram, we do not exhaust their properties; these properties in fact deal with only forty-five of their intersections and ignore the remaining sixty, which are of equal importance in the case of the quartic curve, but coalesce by tens in the Hexagram: for instance, it is easy to verify the statement that the intersections of ab with ac, of ad with ae, and of bc with de, are collinear in the case of the quartic curve; the same statement is nugatory in the case of the Hexagram, and is false in the case of the double tangents of the six-cusped sextic curve spoken of above. This matter will be considered more fully hereafter.

The relations between the different families of sixteen double tangents, of which the quartic has sixty-three, and other details, which possibly would repay investigation, must be passed over altogether. In the case of a quartic having a node, the properties of the Hexagram belong to any fifteen of the sixteen double tangents: they must belong also in a modified form to certain sets of lines composed partly of double tangents and partly of tangents from the node.

§ 4.

Extension to space of four dimensions.

As the outcome of the investigations of §§ 1, 2, we may supplement Cremona's theorem, — that all Veronese's properties of the Hexagram follow intuitively from the geometrical nature of any projection of a three-dimensional figure composed of fifteen lines which lie by threes in fifteen planes, — by the statement that 'All Veronese's results are established almost instantaneously by analytical methods in the case of any projection upon a plane of a three-dimensional figure, of which the nucleus is a set of fifteen lines satisfying a family of fifteen equations such as

$$(1)\quad \begin{cases} x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = 0, \\ \text{when} \\ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0: \end{cases}$$

and which consists in its entirety of the two families of planes $x_1 \pm x_2 = 0$, and their intersections.' But having thus as it were translated Cremona's geometrical theorem into the language of analysis, we cannot fail to see that it may be at once extended and simplified. For a system of six homogeneous coordinates connected by a single identical linear relation should be interpreted as referring to four-dimensional loci; — dealing of course with *descriptive*, to the exclusion of *metrical*, properties.

[A uniform nomenclature for geometry of four-dimensional space has not yet been agreed upon, certain words, (e. g. *plane* and *surface*), being used by different writers in different senses. I shall here call a flat space of n dimensions an R_n; so that a straight line is the same as an R_1; the term a *plane* will always be used to denote an R_2, and the term a *space* without qualification as to the number of dimensions to denote an R_3. Curved loci of one and two dimensions are called *curves* and *surfaces* respectively, and the name *variety* is here restricted to curved loci of three dimensions: thus, in an R_4, varieties, surfaces and curves are determined respectively by one, two, and three relations among the coordinates of their points. Elementary properties of an R_4, e. g. that in it two planes intersect in a single point, or that three given lines are met by one and only one other line, will be assumed; but references to *Veronese's Fondamenti di Geometria, Padua*, 1891, Part II, Book I; to the memoir by the same author, *Math. Annalen*, XIX, p. 161; and to *Whitehead, Universal Algebra, Cambridge*, 1898, Book III, may not be out of place].

Equations (1) then must be interpreted as referring to loci in an R_4; they represent in fact fifteen planes which lie by threes in the fifteen spaces $x_1 + x_2 = 0$. To assume, as we have done hitherto, that the six coordinates (x) are connected by a second identical linear relation which we do not need to specify, is to confine our attention to such parts of the four-dimensional figure as lie in one arbitrary R_3, and is as unscientific as the attempt to realize the nature of a figure in space by discussing its section by a single arbitrary plane: mereover analytical methods bring into very clear prominence the fact that those properties of Cremona's three-dimensional family of fifteen lines of which use has been made, are precisely the properties which it possesses in virtue of being the section by an R_3 of our four-dimensional family of fifteen planes. Again the whole of the four-dimensional figure must be derivable from a set of six spaces $x_1 = 0, x_2 = 0, \dots x_6 = 0$, whose equations satisfy $\Sigma(x_r) = 0$;

in other words it springs from an extremely simple source, six spaces chosen at random in an R_4: for all these reasons the four-dimensional figure is to be preferred.

Since Cremona proceeds to project his family of fifteen lines upon a plane, we now project the four-dimensional family of planes (1) upon an R_3 by means of lines drawn through an arbitrary vertex, and thus arrive at a three-dimensional family of fifteen planes, every plane section of which is a family of lines such as Cremona obtained, and such as we have discussed in §§ 1 and 2: to this three-dimensional family of planes belong properties analogous to, but at once simpler, more extensive and more fundamental than those of the families of lines considered in §§ 1 and 2; properties which would be obvious to us intuitively if we could picture to ourselves the figure in R_4; and which might without difficulty be established by pure geometrical reasoning; but which may also be obtained almost instantaneously from equations (1). The investigation of these properties may clearly be accomplished wholly by means of straight lines, planes, and spaces, — loci, that is to say, of the first order; but as before it is convenient to consider certain curved loci in connection with them. Thus the planes (1) are a part of the locus

(2) $$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 = 0, \quad \text{or} \quad \Sigma(x_r^3) = 0,$$

an extremely interesting variety of the third, denoted in what follows by V_3, some of whose properties have been briefly stated in a note by Segre (*Atti d. R. Accad. di Scienze di Torino*, XXII, p. 547—557). When we project the planes (1) upon an R_3, we also (following the procedure of Geiser) construct the lines which pass through the vertex of projection and touch V_3; they intersect the R_3 in a surface touched by the projection of each of the planes (1) at each point of a conic section: the degree of this surface as a rule is six, but is reduced by two in the important special case when the vertex of projection lies on V_3; moreover, just as before, the curious fact presents itself that, while for arbitrary positions of the vertex of projection we have to deal with a family of fifteen planes possessed of a certain set of properties, in the special case the fifteen are joined by a sixteenth which forms with them an absolutely symmetrical system such that any fifteen possess all the previous properties. In the special case the surface is that known as Kummer's quartic surface, which has sixteen singular points and sixteen singular tangent planes: a plane section of this surface and its singular planes obviously consists of a quartic curve and sixteen of its double tangents. We now learn that the sixteen form a set such as we noticed in § 2, and further that, — (although the statement that any fifteen of them possess all Veronese's properties of Pascal's Hexagram includes almost all that the investigations of Steiner, Hesse and others established concerning such properties of the double tangents of a plane quartic curve, and much beside), — all these properties of the plane curve are in reality consequences of more fundamental theorems concerning Kummer's surface, — which in their turn depend upon the nature of the four-dimensional figure.

In the next section the foregoing and kindred matters will be considered in the simpler and more convenient shape in which the principle of duality enables us to exhibit them. From six points chosen at random in a space of four dimensions a family of fifteen lines which meet by threes in fifteen points, reciprocals of the planes (1), will be derived: our intuitive conception of the nature of this four-dimensional figure suggests a series of properties of the fifteen points in space where the fifteen points in space where the fifteen lines cut any R_3, and from these we can deduce properties of any family of fifteen points in a plane which are the projections of such a family of fifteen points in space: the plane families of points will prove to be the reciprocals of the families of lines discussed in § 2, and their properties to be identical with the reciprocated form of the long list of theorems gradually worked out by mathematicians in the particular case of Pascal's Hexagram, which form the substance of Veronese's memoir.

The reciprocal of Segre's cubic variety V_3 is a variety of the fourth order V_4, of which the fifteen lines spoken of above are double lines: considering the section of this by an R_3, we see that the three-dimensional family of fifteen points mentioned in the last paragraph are double points of a quartic surface, and form a configuration studied by Kummer and others; see Salmon-Fiedler, p. LVII. Should the section be made by an R_3 which touches V_4, the surface has a sixteenth node, and is again Kummer's quartic surface. In order to arrive at the reciprocated form of Pascal's Hexagram a two-fold limitation of the generality is necessary: we must first see that the section of the four-dimensional figure is made by an R_3 which touches V_4, and then take the point of contact as vertex of projection when we project the fifteeen points on a plane.

§ 4.

The figure formed by six points in an R_4.

[In equations (1) and (2) we have made use of a system of six homogeneous coordinates (x) which were connected by an identical linear relation $\Sigma(x_r) \equiv 0$; clearly therefore, although the coordinates of any known point are determined without ambiguity, yet when we work with

$u_1, u_2, \ldots u_6$, coordinates of spaces, where $\Sigma(u_r x_r) \equiv 0$, each coordinate (u) is liable on account of the relation $\Sigma(x_r) \equiv 0$ to be increased by the same quantity and we can only expect relations among the *differences* of the coordinates (u): (to explain the matter more satisfactorily, we must consider the coordinates (x) as referring to an R_4 which forms a part of an R_5). In what follows we discuss for the most part the figure reciprocal to the foregoing, so that coordinates of spaces (u) are definite and connected by the relation $\Sigma(u_r) \equiv 0$; while coordinates of points (x) may all be increased by the same quantity without affecting the equations in which they occur: it is however allowable to impose a condition $\Sigma(x_r) \equiv 0$ on the coordinates (x) should it seem desirable, and we shall always suppose this done].

Let it be agreed to denote under the title *Hexastigm* the figure composed of six points or *vertices* 1, 2, 3, 4, 5, 6 chosen at random in an R_4; the fifteen lines or *edges* 12, 13, . . , joining each two vertices; the twenty planes or *faces* 123, 124, . . . and the fifteen spaces 1234, 1235, . . . determined by each set of three or four vertices respectively. The face 123 is said to be opposite to the face 456, and the edge 12 to the face 3456; three edges such as 12, 34, 56 will, by a slight extension of the meaning of the term, be described as three opposite edges of the Hexastigm. To the fifteen points of intersection of any edge with the opposite space I give the name *Cross-points* of the Hexastigm, denoting by P_{12} the Cross-point which lies on the edge 12. Since the cross-points P_{12}, P_{34}, P_{56} of three opposite edges lie each in the three spaces 3456, 5612 and 1234, they are collinear; in fact they lie on the unique line which intersects the three apposite edges 12, 34, 56: such a line I call a *Transversal* of the Hexastigm.

Theorem. *The fifteen transversals of a Hexastigm are a family of fifteen lines which meet by threes in fifteen points, the Cross-points of the Hexastigm.*

The point of the edge 12 which with the cross-point P_{12} divides the edge harmonically is denoted by Q_{12} and is called a harmonic point of the Hexastigm. It has been tacitly assumed that no five vertices of the Hexastigm lie in an R_3, (and *à fortiori* that no four are coplanar, no three collinear, and no two coincident); we may therefore take $u_1 = 0, u_2 = 0, \ldots u_6 = 0$, to be the equations of the six vertices 1, 2, 3, 4, 5, 6, and $\Sigma(u_r) \equiv 0$ to be the relation connecting them: we then arrive at the following equations for determining the above mentioned points, lines, etc.

Vertex 1; $u_1 = 0$; or $x_2 = x_3 = x_4 = x_5 = x_6$:
Edge 12; $u_1 = u_2 = 0$; or $x_3 = x_4 = x_5 = x_6$:

Face 123;	$u_1 = u_2 = u_3 = 0$; or $x_4 = x_5 = x_6$:
Space 1234;	$u_1 = u_2 = u_3 = u_4 = 0$; or $x_5 = x_6$:
Cross-point P_{12};	$u_1 + u_2 = 0$; $x_1 = x_2$; $x_3 = x_4 = x_5 = x_6$:
Harmonic point Q_{12};	$u_1 = u_2$; $\frac{1}{2}(x_1 + x_2) = x_3 = x_4 = x_5 = x_6$:
Transversal P_{12}, P_{34}, P_{56};	$u_1 + u_2 = u_3 + u_4 = u_5 + u_6 = 0$; or $x_1 = x_2$; $x_3 = x_4$; $x_5 = x_6$.

Since it has been agreed to impose the condition $\Sigma(x_r) \equiv 0$, the coordinates of the harmonic points Q_{12} are

$$x_1 + x_2 = 0;\ x_3 = x_4 = x_5 = x_6 = 0.$$

Consider the parts of the Hexastigm which lie in one of its spaces, for example the space 1234. As the simplest way of describing the well known three-dimensional figure formed by them, it may be said that under special circumstances the middle points of the edges of the tetrahedron 1234 are cross-points of the Hexastigm, the infinitely distant points of the edges are harmonic points; the centre of the tetrahedron is the cross-point P_{56}, and the centre of each face is the point where it is intersected by the opposite face of the hexastigm: the general case may be derived projectively. Thus we see

P_{12}, P_{13}, Q_{23}, are collinear;

Q_{12}, Q_{13}, Q_{23}, are collinear;

$P_{12}, P_{23}, P_{34}, P_{41}, P_{56}, Q_{13}, Q_{24}$, are coplanar;

$P_{12}, P_{13}, P_{14}, Q_{23}, Q_{34}, Q_{24}$, are coplanar;

$Q_{12}, Q_{13}, Q_{14}, Q_{23}, Q_{34}, Q_{24}$, are coplanar;

and other similar results, which the above equations-verify immediately. It further appear that ten harmonic points such as $Q_{12}, Q_{13}, Q_{14}, Q_{15}, Q_{23}, Q_{24}, Q_{25}, Q_{34}, Q_{35}, Q_{45}$, lie in an R_3, whose equation reduces, in virtue of the relation $\Sigma(x_r) \equiv 0$, to $x_6 = 0$: the harmonic points are therefore the points of intersection, four by four, of six spaces

$$x_1 = 0,\ x_2 = 0, \ldots x_6 = 0.$$

But these are polars of the six vertices of the hexastigm with respect to an imaginary variety of the second order, $\Sigma(x_r^2) = 0$, or $\Sigma(u_r^2) = 0$; the harmonic point of any edge is therefore the pole of the opposite space of the hexastigm, and the figure is self-reciprocal with respect to this imaginary quadric variety.

The ten spaces such as

$$x_1 + x_2 + x_3 = x_4 + x_5 + x_6$$

form a remarkable family whether looked on as a separate configuration or as a part of the whole figure: each meets nine edges of the hexastigm in their cross-points and the remaining six edges (which lie in two opposite faces of the hexastigm) in their harmonic points; and each contains six transversals of the hexastigm. The reciprocal family of ten points in the figure derived from six arbitrarily chosen spaces $x_r = 0$, are the double points of Segre's cubic variety

$$\Sigma(x_r^3) = 0, \quad \Sigma(x_r) \equiv 0.$$

At present the only means of distinguishing any one transversal from its fellows is to mention the edges met by it: but by adapting yet another well-known result to our purpose we obtain a simple notation for the different transversals, which is in harmony with the notation of §§ 1 and 2. For we can in six distinct ways select a set of five transversals which meet all fifteen edges of the hexastigm, each transversal being a constituent of two such sets: if then the sets be denoted by letters *a*, *b*, *c*, *d*, *e*, *f* a convenient notation for the transversal which belongs to set *a* and set *b* is the symbol *ab*. In the appended table each transversal is given in the new notation, and after it the edges which it meets

ab	12, 34, 56	*bc*	16, 24, 35	*ce*	14, 23, 56
ac	13, 25, 46	*bd*	15, 23, 46	*cf*	15, 26, 34
ad	14, 26, 35	*be*	13, 26, 45	*de*	16, 25, 34
ae	15, 24, 36	*bf*	14, 25, 36	*df*	13, 24, 56
af	16, 23, 45	*cd*	12, 36, 45	*ef*	12, 35, 46

Two transversals, as for example *ab*, *cd*, whose symbols do not contain the same letter, intersect in a cross point of the hexastigm. Let us now consider briefly the properties of the Hexastigm which lead to the properties (α), (β), (γ), (δ), (ε) of § 1. As before I apply to each line and plane the name of the discoverer of the corresponding locus in Pascal's Hexagram.

(α) Through each of two cross points P_{12}, P_{13}, pass three transversals, *ab*, *cd*, *ef* and *ac*, *be*, *df* respectively: each of the former intersects one of the latter, (viz. in one of the cross-points P_{45}, P_{56}, P_{46}) and in therefore coplanar with it: hence the three planes which contain respectively the transversals *ab* and *fd*, *bc* and *de*, *ef* and *ca* pass through the Pascal line joining the cross points P_{22} and P_{13}.

(β) Three Pascal lines which join the cross points P_{12}, P_{13}, P_{14}, lie in one of sixty Kirkman planes; and the Pascal lines and Kirkman planes fall into six figures of ten lines and ten planes. To obtain such a figure, we join the cross-points P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, of five concurrent edges of the Hexastigm in all possible ways by lines and planes; or we may equally well derive it from the intersections of spaces which contain four of these cross-points.

(γ) Three Pascal lines P_{12}, P_{13}, P_{23}, lie in one of twenty Steiner planes, identical with the faces of the hexastigm, and the Steiner-Plücker lines in which the Steiner planes intersect by fours are identical with its edges.

(δ) The Kirkman planes lead without difficulty to the Cayley-Salmon lines ($u_4 = u_5 = u_6$) and Salmon planes $u_3 = u_4 = u_5 = u_6$.

(ϵ) Of the six transversals which lie in the space

$$x_1 + x_2 + x_3 = x_4 + x_5 + x_6 = 0,$$

viz. *ad*, *ae*, *de*, *bc*, *bf*, *cf*, each of the first three intersects each of the second three: they are therefore generators of a quadric surface.

In a discussion of the four-dimensional figure alone it would be unnecessary to mention properties so obvious as these; it is however worth noting that in this four-dimensional figure Cayley-Salmon points and Salmon planes are the polar reciprocals of the Steiner planes and Steiner-Plücker lines with respect to the quadric $\Sigma(x_r^2) = 0$: the connection is lost when we return to three or two dimensions: and again that the spaces mentioned in (ϵ) are in reality of far higher importance than appears above. A discussion of the properties of Segre's cubic variety, or of the no less interesting reciprocal variety of the fourth order

$$\{\Sigma(x_r^2)\}^2 = 4\Sigma(x_r^4); \quad \Sigma(x_r) \equiv 0;$$

does not fall within the scope of this sketch: I hope soon to investigate the matter elsewhere. Such a discussion would shew how it is that the families of fifteen points in space derived from the transversals of a hexastigm by section with an R_3, or the plane families which are the projections of these, present themselves also in connexion with certain curved loci, (and would very probably throw light on the nature of these loci in the case of the quartic surfaces with fifteen or sixteen nodal points). But it is, I think, clear that this, although the historical, is not the best or most scientific way of approaching families of points or lines or planes which possess they properties of the Hexagram. That families of points endowed with these properties exist, both in a plane and in space, is a fact that should be recognized at the outset of descriptive geometry. Just as from the axioms of three-dimensional geometry we deduce the necessity of the existence of homologous triangles from considering the figure formed by five points in space, so here from the axioms of geometry of four dimensions we have considered the nature and properties of a Hexastigm and its transversals in space of four dimensions, and infer that in space of three or two dimensions families of points must exist possessing the long series of properties which we have classed together under the title Veronese's Properties of Pascal's Hexagram.*)

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*) Cf. Quarterly Journal of Pure and Applied Mathematics, Vol. XXXI, pp. 125—160 (1899).