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ON THE FIGURE OF SIX POINTS IN SPACE OF FOUR DIMENSIONS.

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THE history of the development of the idea that light is thrown on the theorems of projective geometry by the conception of a geometry of a higher number of dimensions affords a very striking illustration of Cayley's mathematical insight. The valuable and ever increasing series of investigations concerning the synthetic geometry of n dimensions, which we owe almost entirely to Italian mathematicians, has its source in Veronese's memoir, *Behandlung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen durch das Princip des Projicirens und Schneidens*, (*Math. Ann.* XIX., 1882); but, as *Gino Loria* points out in his admirable historical monograph, *Il passato e il presente delle principali teori geometriche*, (Memorie d. R. Accad. d. Scienze di Torino, Series II. Vol. 38), the value of the method of Veronese had been clearly grasped by Cayley more than thirty years earlier and used by him in a paper published in 1846, in Vol. XXXI. of *Crelle's Journal.* (*Collected Works*, Vol. I., p. 317).

In comparison with much that has already been effected, the aim of the present paper is a modest one; viz. to consider a very simple figure in four-dimensional space in more detail than has apparently yet been done, and than is possible in the case of the analogous figure in n dimensions; and to derive by projection and section certain figures in space or in a plane and their properties. It may be said that if $n+1$ or more points be taken in a space of n dimensions, and a line or plane or other locus of the first order determined by each combination of the points, two, three,......n at a time, the nature of the configurations derived from this by projection or section is fully treated in Veronese's memoir. Three points in a plane, or four in space, or $n+1$ in space of n dimensions, give us nothing beyond such a system of loci of the first order; the smallest number of points in space of n dimensions which give anything more, or which form a figure having any properties of its own (other than axiomatic ones) is $n+2$. Now from four points in a plane we derive the one-dimensional theory of involution and harmonic section, and from five points in space the two-dimensional principle of homology, each as an intuitive consequence of the fundamental axioms of geometry. What, if anything, can we infer by similar means from the conception of a figure formed by six points in space of four dimensions? The result of the inquiry, though not of the fundamental and indispensable nature of the foregoing cases, is yet sufficiently remarkable; we find that a necessary consequence of the conception is that families of fifteen lines must exist in a plane which possess all the long list of properties that have slowly accumulated round the figure formed by joining six points of a conic, commonly known as Pascal's Hexagram. The fifteen lines do not in general join six points, and it is necessary to investigate their nature by analytical methods: they prove to be in most cases of a class already familiar to mathematicians in connexion with curved loci of the fourth order; and the discovery of a means of obtaining them by linear methods is therefore a simplification. Incidentally too we are led to regard some theorems on quartic curves, cubic surfaces, and quartic surfaces having fifteen or sixteen nodes, from a new point of view, viz. in their connection with the four-dimensional figure.

There is at present no recognised system of nomenclature for loci in hypergeometry, the names *plane* and *surface* for example being employed in different senses by different writers. In what follows I shall have to deal almost exclusively with descriptive properties of space of four dimensions, and shall adopt the usage of Italian mathematicians, Veronese, Segre and others;—denoting by S_n a space of n dimensions, and, when four-dimensional loci are under consideration, applying the term a *line* to an S_1, the term a *plane* to an S_2, and the term a *space* (without specification of the number of dimensions) to an S_3. The names *curve* and *surface* refer respectively to curved loci of one and two dimensions, and the word *variety* is confined to curved loci of three dimensions. Thus hereafter, when analytical methods are introduced, loci in S_4 the coordinates of whose points satisfy one or two or three relations, are respectively varieties, surfaces, and curves; but should the relations be all of the first degree they are respectively spaces, planes and lines. Elementary properties of S_4, such as that two planes have as a rule only one common point, and that one and only one line meets three given lines, will be assumed: they are investigated in Veronese's *Fondamenti di Geometria*, (Padua, 1891), Part II., Book I., pp. 457-500: (German translation by Schepp, entitled *Grundzüge der Geometrie* etc., Leipzig 1894).

Section I.

On the nature of a Hexastigm.

Let six points denoted by symbols 1, 2, 3, 4, 5, 6, be chosen at random in a space of four dimensions; excluding exceptional cases, we assume that no five of them lie in an S_3, and *à fortiori* that no four are coplanar, no three collinear and no two coincident. Each pair of these points determines a line, each set of three a plane, and each set of four a space, and to the figure thus constituted I give the name *Hexastigm*, (following Townsend, *Modern Geometry*, Dublin, 1863). The foundation of a Hexastigm is the set of six fundamental points or *vertices*, 1, 2, 3, 4, 5, 6 ; it comprises in addition fifteen lines or *edges*, 12, 13,...... ; twenty planes or faces, 123, 124,...... ; and fifteen spaces 1234, 1235...... Two faces such as 123, 456, are said to be opposite to each other, and the edge 12 is said to be opposite to the space 3456 ; and, by a slight extension of the meaning of the word, three edges such as 12, 34, 56, are described as three opposite edges of the Hexastigm.

The section of the Hexastigm by a space, that is to say the figure formed by those parts of the Hexastigm which lie in an arbitrarily chosen S_3, consists of fifteen points, twenty lines, and fifteen planes, derived respectively from the edges, faces and spaces of the Hexastigm. Through any one of the fifteen points, (for example that derived from the edge 12), pass four of the twenty lines, (viz. those derived from the faces 123, 124, 125, 126), on each of which lie two more of the fifteen points, (derived from the edges 13, 14, 15, 16, and 23, 24, 25, 26). Consider now the two tetrahedra of which these eight points are vertices; corresponding vertices lie on four concurrent lines, as we have just seen; corresponding edges meet in six coplanar points, and corresponding faces in four coplanar lines, and the complete system of fifteen points, twenty lines and fifteen planes is thus accounted for. We have in fact obtained, by a method clearly capable of further development, (which we owe to Prof. Veronese), a very natural and simple analogue in space of the plane figure of two perspective triangles and their axis of homology. The plane sections of the three-dimensional figure formed of the lines and planes determined by five arbitrarily chosen points consist of ten lines and ten points, and can be resolved in ten distinct ways into a pair of perspective triangles and their axis of homology: the *space-sections* (if I may be allowed to coin a convenient word) of the four-dimensional figure formed of the lines, planes, and spaces determined by six arbitrarily chosen points may be resolved in fifteen distinct ways into two perspective tetrahedra and their plane of homology. It may further be pointed out that a plane figure may be derived from that of the two perspective tetrahedra either by projecting it on a plane or by cutting it by a plane: that derived by the latter method consists of twenty points which lie by fours on fifteen lines, and may be resolved in twenty different ways into a set of three perspective triangles and their three concurrent axes of homology. But before discussing such matters further it is best to study the nature and properties of a Hexastigm in S_4 more fully. For analogues in S_n see Veronese, *Fondamenti di Geometria*, Part II., Book II., Chap. II., pp. 550-561, but specially §2. p. 558: see also the memoir in *Math. Ann.* XIX., and, for a different method of investigation, Whitehead, *Universal Algebra*, pp. 139-142.

Returning to the consideration of the Hexastigm, we notice that each pair of opposite faces has in common a single point, and each edge intersects the opposite space in a single point. The latter family of fifteen points will be called *diagonal points* of the Hexastigm, the diagonal point which lies on the edge 12 being denoted by P_{12}, and so for the other edges. A second point Q_{12} is taken on each edge, viz. that which with the diagonal point divides the edge harmonically; Q_{12} will be called the *harmonic point* of the edge 12. The point common to two opposite faces 123 and 456 is written indiscriminately P_{123} or P_{456}, but is of minor importance. In any selected face of the Hexastigm, for example the face 135, we have now, besides the triangle 135, three diagonal points, P_{35}, P_{51}, P_{13}, and three harmonic points Q_{35}, Q_{51}, Q_{13}, upon its sides 35, 51, 13 respectively; and also the point P_{135} where our selected face is intersected by the opposite face 246. The last point is the intersection of three lines which join the vertices of the triangle 135 to the diagonal points of the opposite sides; and so by a well known property of the triangle the following four sets of three points are collinear; Q_{35}, P_{51}, P_{13}; P_{35}, Q_{51}, P_{13}; P_{35}, P_{51}, Q_{13}; Q_{35}, Q_{51}, Q_{13}.

Theorem. The fifteen diagonal points lie by threes on fifteen straight lines. For clearly the diagonal points of three opposite edges of the Hexastigm (such as 12, 34, 56) lie each of them in the three opposite spaces of the Hexastigm (3456, 5612, 1234) and are therefore collinear. This line is the only line which intersects each of the three opposite edges of the

Hexastigm and will be called a *transversal* of the Hexastigm: the three transversals which meet any one edge all meet it in its diagonal point.

For the sake of realizing clearly the nature of the Hexastigm, its diagonal points, harmonic points, and transversals, it is convenient to consider separately such parts of the complete four-dimensional figure as fall within some chosen three-dimensional space. Thus in one of the spaces of the Hexastigm determined by four vertices 1, 2, 3, 4, are contained a tetrahedron 1234 whose vertices, edges and faces are vertices, edges and faces of the Hexastigm; the diagonal points, harmonic points, etc. which belong to those edges and faces, and, in addition, P_{56} the diagonal point of the opposite edge 56. In order to describe the configuration in the simplest way I allow myself to make use of the language of metrical geometry, and describe a particular case (from which the general case may be derived projectively) thus:—under special circumstances the point P_{56} is the centre of mean position of the points 1, 2, 3, 4; the diagonal points P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34} are the middle points of the edges of the tetrahedron 1234; the harmonic points are at infinity; the centres of the faces of the tetrahedron are the points where they are met by the opposite faces of the Hexastigm, and the lines which join the middle points of opposite edges of the tetrahedron are transversals of the Hexastigm. Hence, by projection, the following sets of points are in all cases collinear:—

(1) The vertices 1, 2, and P_{12}, Q_{12}.

(2) P_{12}, P_{34}, P_{56}.

(3) P_{12}, P_{13}, Q_{23}.

(4) Q_{12}, Q_{13}, Q_{23}.

And the following are coplanar:—

(5) The vertices 1, 2, 3, and P_{12}, P_{23}, P_{31}, Q_{12}, Q_{23}, Q_{31}.

(6) The vertices 1, 2, and P_{12}, Q_{12}, P_{34}, P_{56}.

(7) P_{56}, P_{12}, P_{34}, P_{14}, P_{23}, Q_{12}, Q_{34}.

(8) P_{12}, P_{13}, P_{14}, Q_{23}, Q_{34}, Q_{42}.

(9) Q_{12}, Q_{13}, Q_{14}, Q_{23}, Q_{24}, Q_{34}.

As a consequence of (9) we learn that the harmonic points of the ten edges which join any five vertices of the Hexastigm lie in an S_3 and that the fifteen harmonic points are the points of intersection four by four of six spaces. Thus, from the family of six random points in an S_4 we have worked round to a family of six spaces; but it would be equally simple to develop the figure from six random spaces of an S_4 and end with a family of six points. The Hexastigm is therefore reproduced by the principle of duality, and in fact will be shewn later to be its own polar reciprocal with respect to a certain imaginary quadric variety. To state this matter more explicitly, the principle of duality establishes a correspondence in the Hexastigm between each vertex and the space containing the harmonic points of the ten edges which join the remaining five vertices; between the harmonic point of any edge and the opposite space; between the diagonal point P_{12} and the space containing $Q_{12}, Q_{34}, Q_{35}, Q_{36}, Q_{45}, Q_{46}, Q_{56}$; between the transversal which meets the edges 12, 34, 56, and the plane containing Q_{12}, Q_{34}, Q_{56}; and so forth.

Again the space which contains the coplanar system of points (7) and the point P_{16} contains of necessity the following nine diagonal points and six harmonic points,—

$$P_{12}, P_{14}, P_{16}; P_{32}, P_{34}, P_{36}; P_{52}, P_{54}, P_{56};$$

$$Q_{35}, Q_{51}, Q_{13}; Q_{46}, Q_{62}, Q_{24}.$$

To construct this three-dimensional figure, (of which the Hexastigm contains ten examples), take any three lines in space and any three others intersecting the former three: the diagonal points lie at the nine intersections of these lines, which are thus transversals of the Hexastigm and must also be generators of a quadric surface: the harmonic points are now easily determined, since Q_{13} is common to the lines which join P_{12} to P_{32}, P_{14} to P_{34}, P_{16} to P_{36}. It will be observed that these ten spaces meet every edge of the Hexastigm either in its diagonal point or its harmonic point: they will be called *cardinal spaces* of the Hexastigm, and constitute an extremely interesting configuration in S_4, whether regarded as part of the Hexastigm or as distinct from it. The principle of duality explained in the last paragraph connects these ten spaces reciprocally with the family of ten points each of which is common to two opposite faces of the Hexastigm, and each of which we agreed to denote by either of two symbols P_{123} or P_{456}. That cardinal space which contains the harmonic points of six edges situated in the opposite faces 123, 456 of the Hexastigm and the diagonal points of the remaining nine edges will be denoted by the symbol $C(123, 456)$. Each cardinal space contains as we have seen nine diagonal points, six harmonic points, and six transversals which are generators of a quadric surface; conversely each diagonal point lies in six cardinal spaces, each harmonic point in four and each transversal also in four. Any two cardinal spaces intersect in a plane,—one of a system of forty-five,—in which lie two intersecting transversals: thus the two spaces

$$C\,(123,\ 456) \text{ and } C\,(124,\ 356)$$

intersect in a plane containing the two transversals on which lie P_{34}, P_{15}, P_{26}, and P_{34}, P_{16}, P_{25}. In this plane the points P_{15}, P_{16}, P_{26}, P_{25}, form a quadrangle whose opposite sides intersect in the points P_{34}, Q_{12}, Q_{56}. The analytical methods of III. will be found of great help in such investigations as this.

Up to this point the only way of describing a particular transversal of the Hexastigm and distinguishing it from its fellows has been to mention the diagonal points through which it passes or the edges which it meets. The following considerations suggest a simpler notation. It will be found that from the fifteen transversals it is possible in six distinct ways to select a set of five which meet all fifteen edges of the Hexastigm: call them set *a*, set *b*, set *c*, set *d*, set *e*, set *f*; each transversal enters into two of these sets, and therefore the symbol *ab* is suitable as a means of representing that transversal which belongs both to set *a* and set *b*. The constitution of the six sets will readily be inferred from the appended tables; in the former of which the new symbol for each transversal is followed by a list of the edges which it meets; and in the latter, after each edge is written a list of the transversals which meet it:—

Table I.

ab	12, 34, 56	*bc*	16, 24, 35	*ce*	14, 23, 56
ac	13, 25, 46	*bd*	15, 23, 46	*cf*	15, 26, 34
ad	14, 26, 35	*be*	13, 26, 45	*de*	16, 25, 34
ae	15, 24, 36	*bf*	14, 25, 36	*df*	13, 24, 56
af	16, 23, 45	*cd*	12, 36, 45	*ef*	12, 35, 46

Table II.

12	*ab, cd, ef*	23	*af, bd, ce*	35	*ad, bc, ef*
13	*ac, be, df*	24	*ae, bc, df*	36	*ae, bf, cd*
14	*ad, bf, ce*	25	*ac, bf, de*	45	*af, be, cd*
15	*ae, bd, cf*	26	*ad, be, cf*	46	*ac, bd, ef*
16	*af, bc, de*	34	*ab, cf, de*	56	*ab, ce, df*

The first column of table I contains the transversals belonging to set a, and the members of other sets may be selected without difficulty when necessary. Three transversals which pass through a diagonal point are represented by symbols such as *ab*, *cd*, *ef*, in which all six letters occur; and generally two transversals do or do not intersect according as their representative symbols do not or do possess a letter in common: in other words two transversals which belong to the same set do not intersect, and two which do not belong to the same set must do so. Six transversals which lie in one of the ten cardinal spaces of the Hexastigm are therefore denoted by symbols such as *bc*, *ca*, *ab*, *ef*, *fd*, *de*; a fact which suggests a second notation for the cardinal spaces, to be used concurrently with that already explained, viz. the symbol $C\,(abc.\ def)$ to denote the above space, which would be written C (145. 236) in the previous system. The connexion between the two notations is shewn in the following table:

Table III.

$C\,(abc.def) = C\,(145.236)$	$C\,(ace.bdf) = C\,(126.345)$
$C\,(abd.cef) = C\,(136.245)$	$C\,(acf.bde) = C\,(124.356)$
$C\,(abe.cdf) = C\,(146.235)$	$C\,(ade.bcf) = C\,(123.456)$
$C\,(abf.cde) = C\,(135.246)$	$C\,(adf.bce) = C\,(125.346)$
$C\,(acd.bef) = C\,(156.234)$	$C\,(aef.bcd) = C\,(134.256)$

[A certain reciprocity (which however is really illusory, and becomes misleading if pursued too far), will be noticed in these tables between the symbols *a* and 1, *b* and 2, *c* and 3, *d* and 4, *e* and 5, *f* and 6: thus the transversal *ce* meets the edges 14, 23, 56, and reciprocally the edge 35 is met by transversals *ad*, *bc*, *ef*, and so throughout: it does not appear that this reciprocity can be followed out to any result of value, but the construction of the foregoing tables is considerably facilitated by it].

Numerous instances of harmonic section and involution will be found in the Hexastigm, so many indeed that the wisest plan appears to be to pass them over for the present, since after the introduction of the methods of analysis such properties are far more easily discerned: for the same reason further investigation of other details is postponed; but it will be well, as a conclusion to this part of the subject, briefly to consider how the Hexastigm may be built up from a different foundation, viz. the family of ten cardinal spaces. In section II. when we turn to the contemplation of the nature of space-sections of the Hexastigm, we shall no longer be able to regard the family of six random points as the basis of the figure, for the points of the four-dimensional figure will be lost: its lines, planes and spaces will however persist in the shape of points, lines and planes in the S_3 by which the section is made, and on this account the suggested change of standpoint from which we view the structure we have raised becomes desirable. It is necessary to retain both notations for the cardinal spaces, the 1, 2, 3,......notation to shew their relations with the edges and faces of the Hexastigm, the *a*, *b*, *c*,......notation to shew their relations with the transversals; constant references must therefore be made to the three tables given above.

Each of the ten cardinal spaces corresponds to one of the ten ways in which the six letters *a*, *b*, *c*, *d*, *e*, *f*, or the six figures 1, 2, 3, 4, 5, 6, can be subdivided into two triads (or sets of three), as the representative symbols shew. From either symbol for any one space the symbol for any of the other nine spaces may be derived by an interchange of two letters or figures, one from each triad: and deleting the interchanged members we obtain a convenient symbol for the forty-five *pairs* of cardinal spaces: thus the pair

$$C\,(123.456) = C\,(ade.bcf) \text{ and } C\,(135.246) = C\,(abf.cde)$$

is represented by either $C\,(13.46)$ or $C\,(bf.de)$. The latter symbol at once informs us that the transversals *bf*, *de* are common to the two spaces; the former that the points Q_{12}, Q_{46}, P_{14}, P_{16}, P_{34}, P_{36}, P_{23} lie in each. The six cardinal spaces which contain the diagonal point P_{24} (the intersection of transversals *ae*, *bc*, *df*) fall into three pairs C (13.56), C (15.36), C (16.35) or C (*ae*.*bc*), C (*bc*.*df*), C (*ae*.*df*); the remaining four spaces contain the point Q_{24} and have the figures 2, 4, in the same triad. On the other hand the transversal *bd*, which meets the edges 15, 23, 46, is cut by two spaces in each of its diagonal points, viz. by the pair C (23.46) = C (*ac*.*cf* in P_{15}, by the pair (C 15.46) = C *af*.*ce*) in P_{23}, and by the pair C (15.23) = C (*ac*.*ef*) in eP_{26}; and it lies in the other four cardinal spaces, of which it is characteristic that the letters *b*, *d* are members of the same triad.

Three cardinal spaces intersect either in a transversal of the Hexastigm, through which a fourth space also passes, or in one of a family of sixty lines which join the diagonal points of two intersecting edges of the Hexastigm: thus the three spaces

$$C\,(123.456) = C\,(ade.bcf);$$

$$C\,(124.356) = C\,(acf.bde);\quad C\,(125.346) = C\,(adf.bce);$$

intersect in the line joining P_{16} to P_{26}, which passes also through Q_{12}. For a reason to be justified later I call these the sixty *Pascal lines* of the Hexastigm; the line just quoted is also common to three planes which contain the following pairs of transversals, *ad* and *bc*, *de* and *cf*, *eb* and *fa*. We are thus led to study the figure formed of lines which join the diagonal points of different edges: if the edges do not intersect, the line is a transversal and contains a third diagonal point; if the edges do intersect, the line is a Pascal line of the Hexastigm and contains a harmonic point. The sixty Pascal lines fall into six sets of ten lines, which may be called set 1, set 2,......set 6, the members of set 1 being the lines which join each two of the five points P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, the diagonal points of the five edges of the Hexastigm which meet in the vertex 1. But in space of four dimensions a figure built up from five points is equally well determined by the five spaces which contain four of the points, just as in space of three dimensions a tetrahedron is equally well defined by its faces or its vertices. The Pascal lines thus lie by threes in the faces of these figures and by sixes in their spaces: they lie also by threes in the faces of the Hexastigm and by twelves in its spaces.

Finally it may be pointed out that not only is the notation for various loci absolutely symmetrical, but that it is legitimate to interchange in any way either two or more of the symbols 1, 2, 3, 4, 5, 6, or again two or more of the symbols a, b, c, d, e, f; in consequence of this symmetry it is only necessary to give a single example of any type of locus that we discover, since the complete system of similar loci may be obtained by such interchanges. Any correspondence between various loci will be shewn by either scheme of notation just as effectively as though it were explicitly defined in geometrical language. With regard to the curious reciprocity in the groupings of the two sets of six symbols, all that is needed in its application to our present purpose is shewn in tables I, II, III.

Section II.

On space-sections of the Hexastigm, and their projections on a plane.

In considering the three-dimensional figure composed of those parts of the Hexastigm which lie in an arbitrary space, (which for brevity I describe as a space-section of the Hexastigm), I shall, as has been already stated, regard as the foundation of the figure the ten planes which are sections of the ten cardinal spaces; I shall speak of them as the ten cardinal planes of the three-dimensional figure. Since the cardinal spaces of the Hexastigm pass by fours through the the transversals, the cardinal planes of a space-section intersect by fours in fifteen points, (which I shall call the *principal* points), each derived from one transversal. With each cardinal plane and each principal point will be associated the same symbol as with the cardinal space or transversal of which it is the section; thus, in the a, b, c,......notation, to each principal point is assigned a symbol ab, ac,......ef, and to each cardinal plane a symbol such as $C\,(abc.def)$ in such a way that the points which lie in this plane are ab, bc, ca, de, ef, fd; moreover these six points, being derived from six lines which are generators of a quadric surface, must lie on a conic section. But before attempting to establish properties of the space-section, we must realize more exactly the configuration of the ten cardinal planes; as before I shall allow myself to lapse into the language of metrical geometry when it appears to me that clearness of description is gained by so doing.

To this end we observe that, in their relations to any space and opposite edge of the Hexastigm, as for example the edge 12 and the space 3456, the ten cardinal spaces divide themselves into a set of four spaces and another of six. The former meet the edge 12 in its harmonic point Q_{12}, and have no other common point; the latter meet the edge 12 in its diagonal point P_{12} which is also the intersection of the three transversals *ab*, *cd*, *ef*, situated in the space 3456; and two of them contain *cd* and *ef*, two contain *ef* and *ab*, and two contain *ab* and *cd*: moreover each of the other twelve transversals lies in two out of the set of six cardinal spaces and two out of the set of four. It follows that, in a space-section of the Hexastigm, if we select the triangle of principal points *ab*, *cd*, *ef*, two cardinal planes will be found to pass through each of its sides. Thus these six planes form a figure to which, in the special case when *ab*, *cd*, *ef* are at infinity, we should apply the title parallelepiped, and which we may describe in other cases by the phrase a *projected parallelepiped.* The other twelve principal points lie one on each of the twelve edges of the parallelepiped; they lie also two by two on the edges of the tetrahedron formed by the remaining four cardinal planes, the principal points which lie on two opposite edges of the parallelepiped being on the same edge of the tetrahedron. We deduce that the vertices of the tetrahedron are upon the four diagonals of the parallelepiped. This arrangement of the ten cardinal planes is a very convenient one to bear in mind: not only is it easy to picture mentally but it lends itself also to the construction of a model of the cardinal planes and principal points, either simply by marks on the edges of a rectangular box, or better by means of a wire framework in the form of the edges of a parallelepiped, with silk threads passed through holes bored one in each edge at its principal point. It will appear that none of the twelve points need lie on the edges produced, a matter of no small importance in making such a model. It must however always be remembered that the cardinal planes form an absolutely symmetrical system; that no plane or pair of planes can possess any descriptive property which is not possessed equally by every plane or pair of planes of the system. Fig. 2 is copied from such a model, and may be described in the following manner:—

Let SP', SQ', SR', be three concurrent edges of a parallelepiped; $S'P$, $S'Q$, $S'R$, the opposite edges: O the point of concurrence of the diagonals PP', QQ', RR', SS'; p, q, r, s, arbitrary points on these diagonals,—(for compactness it is best to take them upon OP', OQ', OR', OS' respectively). Then the faces of the parallelepiped and of the tetrahedron p, q, r, s shew the configuration of ten cardinal planes of a space-section of a Hexastigm in a (projectively) quite general form; and the principal points are on the edges of the parallelepiped either at infinity or at the points where the edges of the tetrahedron meet them, the following being one of the many possible schemes:—

ce on $Q'R$ and qr; *df* on $R'Q$ and qr;
cf on PS' and ps; *de* on SP' and ps;
ea on $R'P$ and rp; *fb* on $P'R$ and rp;
eb on QS' and qs; *fa* on SQ' and qs;
ac on $P'Q$ and pq; *bd* on $Q'P$ and pq;
ad on RS' and rs; *bc* on SR' and rs;

besides which there lie at infinity,

ab on SP', PS', QR', RQ'; *cd* on SQ', QS', RP', PR';
ef on SR', RS', PQ', QP',

[Some new facts present themselves to our notice in this figure. We observe that the points which lie on the sides of each of the four skew hexagons that we can form of the edges of the parallelepiped, *e.g.* $PQ'RP'QR'P$, lie in a cardinal plane: again we see that the three principal points of three edges which meet in a point form a triangle whose sides are parallel to those of the triangle formed by the principal points of the three opposite edges: these properties however, stated in their projected form, will be found included in the following list. One warning also is needed; fig. 2 is drawn according to the conventional system of perspective used in mathematical diagrams; *i.e.* the eye of the reader is supposed to be indefinitely remote, and yet to see the figure as of finite size, so that the diagram is an orthogonal projection of the configuration in space. But the eye being at infinity is situated in the plane which contains the principal points *ab*, *cd*, *ef*. Thus a mental picture, called up by fig 2, of a three-dimensional configuration of lines, points, and planes, is (projectively) quite general: regarded as a diagram of the projection of this three-dimensional configuration, fig. 2 represents a special case, since in it the points *ab*, *cd*, *ef* have been so projected as to become collinear].

Properties of a space-section of a Hexastigm.

We proceed to make a list of a few of the more important properties of the space-sections of a Hexastigm, illustrating them by references to the foregoing figure. It will be found occasionally that the means of arriving at some line or plane or space in the four-dimensional figure ceases to be available, and a new construction must be devised. It is established, as a consequence of our conception of space of four-dimensions, or of our axioms concerning it, that *There exist in space of three dimensions families of fifteen points which possess the following properties:—*

(α) With each one of the fifteen points may be associated one of the fifteen pairs that can be formed with six symbols, in such a way that any symmetry or correspondence between certain sets of the points is effectively shewn by symmetry or correspondence among the associated symbols; an important consequence of this theorem being that, if we agree to denote each point by the associated pair of six selected symbols, we may, by interchange of the six symbols, alter the representative symbols of different points of the family without affecting the truth of any statement of a property possessed by the points.

Let it be agreed that the fifteen points be called principal points, and that the letters *a*, *b*, *c*, *d*, *e*, *f*, be chosen as the six symbols: each of the fifteen points is denoted one of the symbols, *ab*, *ac*,......*ef*.

(β) Six points *ab*, *bc*, *ca*, *de*, *ef*, *fd*, are coplanar and lie on a conic section: there are in all ten such planes, called cardinal planes, four of which pass through each principal point.

(γ), The intersection of three cardinal planes is either a principal point, through which a fourth cardinal plane will also pass, or one of a set of sixty points, called *Pascal* points: on the line of intersection of any two cardinal planes lie two principal points and four Pascal points; for example the line SP' in figure 2 passes through two principal points *ab* and *de* and four Pascal points viz. S, P', and its intersection with *pqr*, *qrs*. Each Pascal point, for example S, is the intersection of three lines of intersection of two cardinal planes, SP', SQ', SR', which join *ab* to *de*, *bc* to *ef*, and *cd* to *fa*. Each Pascal point may be described as the intersection of lines joining opposite vertices of a skew hexagon whose vertices are *ab*, *bc*, *cd*, *de*, *ef*, *fa;* and therefore each corresponds to one of the sixty reversible cyclical arrangements of the symbols *a*,*b*,*c*, *d*,*e*,*f*.

(δ) The Pascal points fall into six sets of ten points, the members of each set constituting the vertices of a figure formed by five planes. The ten points of a set therefore lie by threes in ten lines and by sixes in the five planes. An illustration of such a set is furnished by the four points P, Q, R, S, of figure 2, and the six points where the lines PS, QS, RS, QR, RP, PQ, meet ps, qs, rs, qr, rp, pq, respectively: the tetrahedra $PQRS$ and $pqrs$ are perspective; the lines Pp, Qq, Rr, Ss, joining corresponding vertices meet in O; the six points of intersection of corresponding edges just mentioned lie in a plane which with the faces of the tetrahedron $PQRS$ forms the set of five planes referred to.

(ϵ) The Pascal points lie also by threes on twenty lines and by twelves in fifteen planes, (derived respectively from the faces and spaces of the Hexastigm), which form the figure described at the beginning of I., resoluble in fifteen ways into a pair of perspective tetrahedra. In figure 2, the set of planes derived from the spaces of the Hexastigm is made up of the plane at infinity, the six planes which pass through two opposite edges of the parallelepiped, and eight others each containing the principal points of three concurrent edges: of these eight, the four which pass through the principal points of edges meeting in P, Q, R, S, form one tetrahedron whose vertices lie on PP', QQ', RR', SS' respectively, and the remaining four form a second tetrahedron whose vertices lie also on these lines and whose faces are parallel to those of the former: the two tetrahedra have O as centre of perspective and the plane at infinity as plane of homology. The twelve Pascal points which lie at infinity are intersections of the three lines at infinity of the faces of the parallelepiped with the faces of $pqrs$.

(ζ) Each of the six sets of ten Pascal lines in (δ) was determined by section of the edges of a figure defined by five diagonal points of the Hexastigm such as P_{12}, P_{13}, P_{14}, P_{15}, P_{16}: the faces of these six figures intersect by threes in the lines such as Q_{56}, Q_{64}, Q_{45}, and the spaces intersect by twos in the planes similar to that containing Q_{34}, Q_{35}, Q_{36}, Q_{45}, Q_{46}, Q_{56}. But we have seen that the harmonic points are determined fully by the intersection of six spaces; hence, returning to the space-section of the Hexastigm, we infer that there is a certain figure of six planes through each of whose edges pass two faces of the figures (δ), and through each of whose vertices pass three edges of these figures.

The plane figure.

Instead of considering the plane figure derived from the space-section of a Hexastigm by projection, we find it rather more convenient to treat of its reciprocal, that is, a plane section of the figure derived by the principle of duality from that which we have just discussed. The six vertices and ten cardinal spaces of the Hexastigm being both unavailable, we look on the fifteen lines derived from the transversals as the foundation of the plane figure. The reason for describing the lines and points of the figure by the names of various mathematicians will appear shortly, if it is not already recognized.

A necessary consequence of our conception of space of four dimensions is that there exist in a plane families of fifteen lines possessing the following properties:—

(α) Each line may be associated with, and denoted by, one of the fifteen symbols ab, ac,.........ef, (formed by selecting two out of six symbols a, b, c, d, e, f, in all possible ways), in such a manner that the validity of any statement concerning the lines is unaffected by an interchange of these six symbols, and any symmetry or correspondence of various sets of the lines is shewn effectively by their representative symbols.

(β) Six lines such as ab, bc, ca, de, ef, fd, touch a conic; there being in all ten such conics, any two of which have two of the fifteen lines as common tangents.

(γ) Corresponding to the different permutations of the six letters a, b, c, d, e, f, (*e.g. acebdf*) we may form sixty hexagons such as ac, ce, eb, bd, df, fa; the three intersections of opposite sides of any such hexagon are collinear; I call these lines the sixty *Pascal* lines.

(δ) The sixty Pascal lines fall into six sets of ten lines, each set forming the well known configuration of ten lines and ten points, which may be resolved in ten different ways into a pair of perspective triangles. The ten Pascal lines of a set thus meet by threes in ten points which I call Kirkman points. That there is a correspondence between each of the sixty Pascal lines and one of the sixty Kirkman points which is a member of the same set, is obvious from the nature of the configuration; for any separate one of the ten points of a set may be taken as a centre of perspective of two triangles, and their axis of homology is the corresponding Pascal line.

(ϵ) The Pascal lines also meet by threes in twenty *Steiner* points which lie by fours on fifteen *Plücker* lines. These form another well known configuration, the projection of the intersections of six planes in space, which may be resolved in twenty distinct ways into three triangles whose vertices lie on three concurrent lines, and the three concurrent axes of homology of each pair.

(ζ) When three Pascal lines meet in a Steiner point their three corresponding Kirkman points lie on one of twenty *Cayley-Salmon* lines, which meet by fours in fifteen *Salmon* points, and form a configuration reciprocal to that in (ϵ).

Now these six theorems, which may be obtained without difficulty from previous results concerning the Hexastigm and its space-sections, have a very familiar form; a reference to the note at the end of Salmon's Conic Sections shews that, if a, b, c, d, e, f, denote six points of a curve of the second degree, the fifteen lines ab, ac,......ef, which join each two of them possess numerous properties included in the above theorems. It would be extremely rash to assume that the fifteen lines ab, ac......ef, which we have derived from the Hexastigm, necessarily join six points of a conic; but it is clearly advisable to see to what extent their properties (intuitive consequences of the nature of a simple four-dimensional figure) are in agreement with those better known results which in the middle half of the present century were accumulated round the celebrated theorem discovered by Pascal more than two hundred and fifty years ago, and still known by his name.

The development of the figure now commonly known as the *Pascal Hexagram* dates from 1828, when Steiner drew attention to the important fact that, from the same six points of a conic section, sixty distinct hexagons can be formed, each with its own Pascal line. During the next fifty years the figure formed of these sixty lines aroused wide interest, Steiner, Plücker, Hesse, v. Staudt, Schöter, Cayley, Salmon, Kirkman, and many others applying themselves to the study of its properties. To dwell in detail on the advances made by each of these mathematicians is superfluous, since the results of their labours have been summed up and extended by Veronese, in a masterly memoir, *Nuovi teoremi sull' Hexagrammum Mysticum*, (*Atti d. R. Accad. dei Lincei*, 1877, Vol. I, Series III, pp. 642-703), which is prefaced by an excellent historical sketch, with full references to the works of earlier writers, and contains proofs not only of all previously known theorems but of a large number of new and original ones. The names used in (γ), (δ), (ϵ), (ζ) are adopted from Veronese's memoir, and the subdivision of the Pascal lines and Kirkman points in (δ) into six sets is the most important of his original contributions to the theory.

The results of a thorough investigation of Veronese's memoir may be stated as follows;—*If we join in all possible ways, by lines, planes, and spaces, the diagonal points and harmonic points of a Hexastigm; take a space-section of the figure so formed; reciprocate it; and take a plane section of the reciprocal; we obtain a plane figure built up from fifteen lines, coextensive with that built up by Veronese for the special case when the fifteen lines join in pairs six points of a conic, together with proofs (intuitive consequences of the nature of the four-dimensional figure) of all his theorems:* (there must of course be exceptions to so sweeping an assertion as this, but they are so few and so trivial that it seems justifiable to ignore them).

By far the most important addition to the subject since the publication of Veronese's memoir is due to Cremona and will be referred to later: so far as I am aware nothing has appeared which renders inadmissible the statement, that, *the existence in a plane of other families of fifteen lines, which possess practically all known properties of the fifteen lines that join six points of a conic, is a necessary consequence of our conception of space of four dimensions;* or more precisely of the axiomatic law that in it, lines, planes, and spaces are determined by two, three, and four points respectively, and are cut by a space of three dimensions in points, lines, and planes. It must be admitted that the later properties of Veronese's memoir become tedious when considered in detail and are of less importance than the earlier results: the properties quoted (α), (β), (γ), (δ), (ϵ), (ζ), carry us as far into the subject of these families of fifteen lines as it seems advisable to penetrate. Far more important than the extension of the long list of elementary geometrical results concerning them is the enquiry as to the nature of these lines in the general case; for the discussion of this I call in the aid of Analytical methods.

The transition from the set of six points in S_4 to the fifteen lines in a plane has been accomplished by three operations, (1) a section, (2) a reciprocation, (3) a section: but the operations of section, reciprocation and projection are commutative, if we allow for the fact that a reciprocation interchanges the other two. Instead of the above process we may make the passage from four to two dimensions by first reciprocating, then taking a section and finally projecting, or in many other ways; the initial and final figures being always the same, but the intermediate ones of quite different types.

Section III.

Analytical Methods.

Whatever be the number of dimensions of the space we are considering, the *coordinates* of its points will be denoted by letters x, X, and the *equations* of its points by letters u, U, with suffixes added; the capitals being used for fixed, the small letters for current coordinates.

In a space of four dimensions S_4, the equations of any six points must be connected by one identical linear relation; in the case of the vertices of the Hexastigm it was stipulated that no five were to lie in a space of three dimensions, and we are therefore at liberty to represent the vertices 1, 2,......6, by equations $u_1 = 0$, $u_2 = 0$, ... $u_6 = 0$, which satisfy the identity

$$u_1 + u_2 + u_3 + u_4 + u_5 + u_6 \equiv 0;$$

or $$\Sigma (u_r) \equiv 0; \ (r = 1, 2, 3, 4, 5, 6).$$

The equations which determine any points, lines, planes or spaces in the Hexastigm now become apparent; the edge 12 is $u_1 = 0$, $u_2 = 0$; the face 123 is $u_1 = 0$, $u_2 = 0$, $u_3 = 0$; and the space 1234 is $u_1 = 0$, $u_2 = 0$, $u_3 = 0$, $u_4 = 0$: the diagonal point P_{12}, common to the edge 12 and the space 3456 is $u_1 + u_2 = 0$, and the harmonic point Q_{12} is therefore $u_1 - u_2 = 0$. The transversal P_{12}, P_{34}, P_{56} is $u_1 + u_2 = 0$, $u_3 + u_4 = 0$, $u_5 + u_6 = 0$; and the cardinal space $C\,(123 \,.\, 456)$ has coordinates

$$u_1 = u_2 = u_3 = -u_4 = -u_5 = -u_6: \text{ etc. etc.}$$

As regards coordinates of points (x), we choose them in the first instance to satisfy the indentity

$$\Sigma (u_r x_r) \equiv 0, \ (r = 1, 2, 3, 4, 5, 6),$$

and make no further condition: on account of the former identical relation $\Sigma (u_r) \equiv 0$, each coordinate x is liable to be increased by the same quantity, and we can therefore only expect to obtain relations among the *differences* of the coordinates x: for example the space 3456 is $x_1 = x_2$, the face 456 is $x_1 = x_2 = x_3$, and the edge 56 is $x_1 = x_2 = x_3 = x_4$: the transversal and the cardinal space quoted above are represented

by $x_1 = x_2$; $x_3 = x_4$; $x_5 = x_6$; and by $x_1 + x_2 + x_3 = x_4 + x_5 + x_6$;

respectively. We have perfect right to impose another condition on the coordinates x but at present there is no advantage gained by doing so.

The case is altered when we come to the harmonic points, which we proved were determined by the intersections of six spaces, viz. $u_2 = u_3 = u_4 = u_5 = u_6$, or $6x_1 = \Sigma(x_r)$ etc. By imposing the condition $\Sigma(x_r) \equiv 0$, the equations of these spaces become $x_1 = 0$, $x_2 = 0, \ldots\ldots x_6 = 0$; not only is the figure self-dualistic as stated in I., but it is actually its own polar reciprocal with respect to the imaginary quadric variety

$$\Sigma(x_r^2) = 0; \text{ or } \Sigma(u_r^2) = 0; \ (r = 1, 2, 3, 4, 5, 6);$$

as may be easily verified: the harmonic point of each edge of the Hexastigm is the pole of the opposite space with respect to this quadric, and the diagonal point is the pole of the space which contains the harmonic point of that edge and of each of the opposite space. Thus, when our coordinates satisfy the identities $\Sigma(x_r) \equiv 0$; $\Sigma(u_r) \equiv 0$; $\Sigma(u_r x_r) \equiv 0$; from the six points $u_r = 0$ we work round to the six spaces $x_r = 0$; and the six spaces would serve equally well as the foundation of the figure. A very important consequence is that, if we discuss fully the space-sections of the complete Hexastigm, we may pass over its projections; for the projections of the figure derived from six points in S_4 are merely reciprocals of the space-sections of the figure derived from six spaces in S_4.

The following are the equations, in both systems of coordinates x and u, of loci connected with the Hexastigm:

$$\Sigma(x_r) \equiv 0; \ \Sigma(u_r) \equiv 0; \ \Sigma(u_r x_r) \equiv 0; \ (r = 1, 2, 3, 4, 5, 6).$$

Vertex 1; $u_1 = 0$; $x_2 = x_3 = x_4 = x_5 = x_6$:

Edge 12; $u_1 = u_2 = 0$; $x_3 = x_4 = x_5 = x_6$:

Face 123; $u_1 = u_2 = u_3 = 0$; $x_4 = x_5 = x_6$:

Space 1234; $u_1 = u_2 = u_3 = u_4 = 0$; $x_5 = x_6$:

Diagonal point P_{12}; $u_1 + u_2 = 0$; $x_1 = x_2$; $x_3 = x_4 = x_5 = x_6$:

Harmonic point Q_{12}; $u_1 = u_2$; $x_3 = x_4 = x_5 = x_6 = 0$.

Transversal line P_{12}, P_{34}, P_{56}; $u_1 + u_2 = u_3 + u_4 = u_5 + u_6 = 0$;

or $\qquad x_1 = x_2$; $x_3 = x_4$; $x_5 = x_6$:

reciprocal to this is the plane containing Q_{12}, Q_{34}, Q_{56}.

Cardinal space $C(123.456)$; $u_1 = u_2 = u_3 = -u_4 = -u_5 = -u_6$;

or $\qquad x_1 + x_2 + x_3 = x_4 + x_5 + x_6 = 0$:

reciprocal to this is the point common to the faces 123, 456.

The symmetry of the two systems of coordinates u and x is so perfect that, while the name Hexastigm is retained to denote the complete series of loci, ranging between the six points $u_r = 0$ on the one hand and the six spaces $x_r = 0$ on the other, it is clearly desirable to recognize as fully as possible the equal claim of the two systems to be regarded as the basis of the figure. We may describe the two as the six-point system and the six-space system respectively, and, just as we have derived from the six-point system diagonal points, harmonic points, transversal lines and cardinal spaces, we derive reciprocal loci from the six-space system, and call them diagonal spaces, harmonic spaces, transversal planes and cardinal points. A diagonal space for example contains the plane common to two spaces of the six-space system and the point common to the remaining four. Certain loci it will be seen appear under different names in the two systems; the harmonic spaces, for instance, of the six-space system are identical with the spaces containing four of the six original points of the six-point system; but the equations we have given will prevent us from overlooking such facts as this. The symmetry is lost when we take a space-section of the Hexastigm but reappears in a less perfect form in the two-dimensional figure derived by projection. The difficulty in the four-dimensional figure is how to connect in a simple geometrical manner the two reciprocal systems—(reciprocation with respect to an imaginary quadric cannot well be used)—and it would be of great use to us for this purpose if a closer connexion between the two systems existed.

We have yet to find the equations of the Pascal lines of the Hexastigm; but in so doing it is best to keep in mind the lines and points they lead to in the plane figure. A Pascal line in the six-point system was determined by two diagonal points such as P_{12}, P_{13}, and its equations are therefore

$$-u_1 = u_2 = u_3 \text{ ; or } x_4 = x_5 = x_6 = x_2 + x_3 - x_1 = -\tfrac{1}{2}\, x_1$$

The Pascal lines which join P_{12}, P_{13}, P_{23} lie in one of the planes 123 of the six-point system, and lead to three Pascal lines in the plane figure which meet in a Steiner point; the Steiner points and Plücker lines in the plane figure being derived from the edges and faces of the six-point system. The Pascal lines and Kirkman points of one of the six sets which Veronese discovered are derived from the edges and faces of the figures formed by joining P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, (or a similar set of diagonal points), in all possible ways; the Pascal line derived from joining P_{12} and P_{13} corresponding to the Kirkman point derived from the plane P_{14}, P_{15}, P_{16}.

Thus a Veronese set of Pascal lines and Kirkman points is derived from either five points or five spaces, *e.g.*

$$u_1 + u_2 = 0\,;\; u_1 + u_3 = 0\,;\ldots\ldots \qquad u_1 + u_6 = 0\,;$$

or $$x_1 + 2\,x_2 = 0\,;\; x_1 + 2\,x_3 = 0\,;\ldots\ldots \qquad x_1 + 2\,x_6 = 0:$$

and it is clear that, when three Pascal lines lie in one of the six-point system, the corresponding planes pass through the reciprocal edge of the six-space system; but the Pascal lines are unfortunately not reciprocals of the planes which lead to the corresponding Kirkman points.

Corresponding investigations in the six-space system are taken for granted. It seems worth while to digress here for a moment in order to point out that it is possible by projective methods to bring six random points in S_4 to a form in which the distance of each two is the same. With the vertices of the Hexastigm arranged thus, the diagonal points bisect the edges, and the figure acquires many beautiful metrical properties. Reduction to this form is not possible in Euclidean space, for the equation $\Sigma\,(u_r^2) = 0$ or $\Sigma\,(x_r^2) = 0$ must represent the Absolute.

Space-sections of a four-dimensional figure.

The coordinates x may be applied at once to space-sections, the sole difference being that they have to satisfy a second identical linear relation, viz. the equation of the space by which the section is made. If U_1, U_2,...... U_6 denote the coordinates of this space, the equations of all loci in the section may be deduced at once from the foregoing formulae in terms of six coordinates x_1, x_2,......x_6, connected by two identical linear relations

$$\Sigma\,(x_r) \equiv 0\,;\;\; \Sigma\,(U_r\,x_r) \equiv 0\,;\;\; (r = 1, 2, 3, 4, 5, 6).$$

But if we wish to use coordinates u, difficulties beset us: on account of the identity $\Sigma\,(U_r\,x_r) = 0$, each of the quantities u_r is liable to be increased by the same multiple of U_r, and thus we shall only be able to interpret equations in u's which are not altered when $u_r + \lambda\, U_r$ is substituted for u_r: we are in just the same case as when at the beginning of this III. we saw that only equations in the differences of the coordinates x were to be expected to arise: as then, we have every right to simplify our equations, if possible, by assuming that the quantities u satisfy a new linear identity. When several equations in u coordinates determine a locus, the space-section by the space $(U_1, U_2, \ldots U_6)$ is found by writing $u_r + \lambda\, U_r$ for

u_r in the equations and eliminating λ. [The matter is more easily explained in its reciprocal form, viz. when we are projecting loci in S_4 upon an S_3 from a centre of projection whose coordinates are $X_1, X_2, \ldots X_6$. Equations in u coordinates here present no difficulty; but, given a locus defined by two or more equations in x coordinates, we first write $x_r + \lambda X_r$ for x_r and eliminate λ: this represents a locus generated by lines which join the point X to each point of the given locus: we may now if we wish, assume a new relation among the coordinates x_1 i.e. specify a particular S_3 as the space on which the projection is made; but it is seldom advisable to do this]. Obviously, in discussing the space sections of the Hexastigm, we must keep as far as possible to x coordinates. As has been stated above, on account of the perfect reciprocity of the Hexastigm, it will not be necessary to discuss its projections into space of three dimensions; for all that concerns them may be obtained by the principle of duality from a space-section, provided the latter be considered in its relation both to the six-point and to the six-space systems: but first we shall turn our attention to a certain variety in S_4 which throws light on the nature of the planes, lines, and points of a space-section, and shews that they have already become to some extent familiar to mathematicians.

On Segre's cubic variety.

Intimately connected with any six-space system in S_4 is a certain variety of the third order, some of whose properties, studied without the aid of analysis, form the subject of a note by its discoverer, Corrado Segre, in vol. XXII of the *Atti della R. Accad. delle Scienze di Torino* 1887. p. 547. This variety, which appears to me to possess far more beautiful properties than any cubic surface in three-dimensional space, is of the fourth class, is rational, has no independent invariant, has the maximum finite number of double points possible in a cubic variety, namely ten. When the equations of the six planes are so prepared that their sum is identically zero, the equation of the variety expresses that the sum of their cubes vanishes. With our six-space system, $x_r = 0$, is associated the Segre's cubic variety, (to be denoted in what succeeds by V_3),

$$\Sigma (x_r)^3 = 0 \; ; \; \Sigma (x_r) \equiv 0 \; ; \; (r = 1, 2, 3, 4, 5, 6,) :$$

the equation may however be also written in ten forms similar to

$$(x_2 + x_3)(x_3 + x_1)(x_1 + x_2) + (x_5 + x_6)(x_6 + x_4)(x_4 + x_5) = 0.$$

The cardinal points ($x_1 = x_2 = x_3 = -x_4 = -x_5 = -x_6$, etc). of the six-space system lie on V_3, and an attempt to determine their tangent spaces shews that each is a double point. The transversal planes, which form a system of fifteen planes situated by threes in the fifteen diagonal spaces lie on V_3, and are part of it: each diagonal space thus cuts V_3 in three planes. Now in S_4 we can draw through any ordinary point of a variety six lines having four-point contact at the point, and in the case of Segre's cubic V_3, these lines must lie wholly on the variety, and must therefore meet each one of the fifteen diagonal spaces in one of the three transversal planes contained in it. Reasoning from this we are able to attach a more definite geometrical significance to the symbols a, b, c, d, e, f of Tables 1, 2, 3, than has hitherto been possible; viz. that the six lines which pass through any point of V_3 and lie wholly on V_3 are of six distinct types, a, b, c, d, e, f, those of type a meet one set of five transversal planes, those of type b another set, and so on: the symbol ab associated with the transversal plane 12, 34, 56 in Table 1, shews that this plane is met by all lines of the types a and b.

Section IV.

On the Pascal Hexagram.

In order to pass from the transversal lines of a six-point system to the plane families of fifteen lines which possess the properties of Pascal's Hexagram, it has been said that three operations are required. One of these is necessarily a reciprocation, but it may be either the first or second or third of the series. In virtue of III., the simplest way of making the transition is to take the reciprocation first, for this merely changes the transversal lines of the six-point system into transversal planes of an equally simple six-space system; projecting a space-section of this family of planes we arrive at the plane figure desired. The derivation of Pascal's Hexagram from the six-space system in this way is the subject of the present section; we have first to consider space-sections of the fifteen planes which formed part of Segre's cubic variety V_3 in general.

Now the section of V_3 by an arbitrary space is a cubic surface of quite general type: for Cremona has shewn (*Math. Annalen* XIII. p. 301) that the equation of a non-singular cubic surface may be reduced to the form

$$\Sigma(x_r^3) = 0; \ \Sigma(x_r) \equiv 0; \ \Sigma(U_r x_r) \equiv 0; \ (r = 1, 2, 3, 4, 5, 6).$$

in thirty six different ways: see *Salmon-Fiedler*, p. 403, section 310. For special positions of the space of section, *i.e.* for special values of $U_1, U_2, \ldots U_6$, the cubic surface may possess singularities; but of such cases I shall consider only one, viz. when the space of section touches V_3 at an ordinary point and the cubic surface therefore has a double point at the point of contact; *Salmon-Fiedler*. p. 412, and foot-note, section 341. The space-sections of the fifteen transversal planes of the six-space system are a family of fifteen lines which lie by threes in fifteen planes, and also lie on the cubic surface; they are a set of fifteen of the twenty-seven lines of the surface such as is left when we omit a double-six; *Salmon-Fiedler*. p. 401, section 308. Schläfli, *Quarterly Journal*, Vol. 2 p. 116. In the special case of section by a space which touches V_3, they are the fifteen lines of the surface which do not go through the nodal point. We thus arrive at a theorem due in part to Cremona, viz.

The plane systems of lines which possess the properties proved for the lines which join six points of a conic are projections of fifteen lines of a cubic surface such as are left when we exclude a double-six. It is necessary to include a statement of the special case, for this arises when the members of the rejected double-six coalesce two by two in six lines through the double point.

Mention was made at the end of II. of a memoir by Cremona: it is to be found in the same volume of the *Atti d. R. Accad. dei Lincei* as that of Veronese; pp. 854-874. On reading Veronese's manuscript Cremona was led to seek another basis for the existence of this vast series of theorems, and found it in the three-dimensional system of lines that lie on a cubic surface having a double point. On such a surface lie six lines which pass through the nodal point, and are generators of the tangent cone, and fifteen others, one in the plane of each pair of the foregoing: by projecting these on a plane, from the nodal point as centre of projection, Cremona obtained fifteen lines joining six points of a conic, and, having established the fact that these lines lie by threes in fifteen planes, shewed that all Veronese's theorems were intuitive consequences. In conclusion, he observes that the projections of any family of fifteen lines which lie by threes in fifteen planes would possess these properties, and that the lines of a cubic surface supply examples of such families, but goes no further, overlooking the fact that Geiser had discussed (*Math. Ann.* I. p. 129) the projections of the lines of a cubic surface, with valuable and well known consequences. Cremona, then, stopped after taking a very important step in the direction of the simplification of the vast figure which Veronese had constructed, in that he shewed how it could be derived, and all its properties established, from a comparative simple figure in space: a discussion of the Hexagram from Cremona's point of view will be found in the *Transactions of the Cambridge Philosophical Society*, vol. XV. p. 207, in which it is pointed out that the complete figure as developed by Veronese was the result, save in a few unimportant details, of projecting the intersections of the two systems of planes analogous to $x_1 \pm x_2 = 0$.

The method followed in the present paper derives the plane figure, and establishes its properties, from one of the simplest possible (descriptive) figures in space of four dimensions, by purely linear methods; it leads us to notice that other systems of coplanar lines and points possess all these properties, of which systems the Pascal Hexagram is an extremely special case: and it will be seen that the transition from four to two dimensions may be made by a different route with no less interesting results. That the true cause for the existence of families of coplanar lines and points, possessed of all Veronese's long category of properties, is to be found in the figures in S_4 cannot be doubted; although we shall find that, as a matter of history, most of these families have been already discovered, and some of their properties obtained, by other means, chiefly in connection with the study of curves and surfaces of the fourth degree. We will now consider how the above fifteen lines in space lead us, under special conditions, to Pascal's Hexagram, and then treat the most general case of a projection of a space-section of the fifteen transversal planes of a six-space system in S_4.

In order to derive from the transversal planes, such as

$$x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = 0,$$

of a system of six spaces $x_r = 0, (r = 1, 2, 3, 4, 5, 6,)$ where $\Sigma (x_r) \equiv 0$, the configuration of lines which join six points of a conic section, let the S_4 in which the spaces lie be cut by a space which touches Segre's variety V_3, $\Sigma (x_r^3) = 0$, in some point K. As explained at the end of III., six lines lying wholly on V_3 pass through K, and these are members of six different families distinguishable by six symbols a, b, c, d, e, f; in such a way that the symbols ab, ac etc., associated with the transversals in Table I, shew us which two of the six lines through K each transversal plane intersects: the first line in that table associates ab with 12, 34, 56; therefore the plane quoted above intersects the lines through K which are of type a and type b. In the space-section by the tangent space at K, V_3 is represented by a cubic surface having a double point at K, and the transversal planes by fifteen lines on the surface; but the lines of V_3 which pass through K persist in the space-section as lines, still distinguishable by letters a, b, c, d, e, f, which lie on the cubic surface, and pass through its double point; and each of the former fifteen lines intersects two of the latter according to a scheme shewn immediately by reference to Table 1.

The analytical formulae are discussed at length in my paper in the *Cambridge Phil. Trans.* to which I have referred. If the coordinates of the point of contact of the space of section with V_3 be denoted by X, we have the following system of equations for a three-dimensional cubic surface endowed with one nodal point K:

Equation of the surface, $\Sigma(x_r^3)=0$; $(r=1, 2, 3, 4, 5, 6)$:

Coordinates of K, *the double point* (X_r):

And the relations which connect the coordinates and constants are

$$\Sigma(x_r)\equiv 0; \ \Sigma(X_r^2x_r)\equiv 0; \ \Sigma(X_r)=0; \ \Sigma(X_r^3)=0.$$

The six lines a, b, c, d, e, f, which lie on the surface and pass through K, are generators of the quadric cone $\Sigma(X_r x_r^2)=0$; but it does not appear that the separate equations can be exhibited in a simple form. The plane through any two, for example c and e, can be found at once, for it meets the surface in a third line ce, whose equation is shewn by Table 1 to be

$$x_1+x_4=x_2+x_3=x_5+x_6=0;$$

the equation of the plane through this line ce, c, e and K is therefore

$$\frac{x_1+x_4}{X_1+X_4}=\frac{x_2+x_3}{X_2+X_3}=\frac{x_5+x_6}{X_5+X_6}.$$

Through each of the fifteen lines ab, ac,......ef, pass three of the planes $x_1+x_2=0$, sections of the fifteen diagonal spaces of the six-space system, according to the scheme shewn in Table 2; and the intersection of two such planes, *e.g.* $x_1+x_2=0$ and $x_1+x_6=0$, where the former contains ab, cd, ef, and the latter af, bc, de, passes through the points of meeting of ab with de, of bc with ef, and of cd with fa.

Project this system on a plane from K as centre of projection: the lines a, b, c, d, e, f, cut the plane in six points of a conic, also denoted by a, b, c, d, e, f, and the lines $ab, ac, \dots ef$ project into lines which join each two of the six, and are therefore naturally still denoted by symbols $ab, ac, \dots ef$. The projection of the line of intersection of the planes $x_1 + x_2 = 0$, $x_1 + x_6 = 0$, still contains the intersections of ab with de, of bc with ef, of cd with fa, and is a Pascal line of the Hexagram.

As regards the equations of loci in the plane figure, we may, from the equations of any line (or curve) in the three-dimensional figure, derive the equation of the plane (or cone) formed by joining each point to K, and thus obtain equations of a system of geometrical loci in space wholly generated by lines through K; practically we do this by writing $x_r + \lambda X_r$ in place of x_r in the equations of the lines (or curve) in space, and eliminating λ. The section of this system by any plane is the projection of the three-dimensional figure on the plane from K as vertex of projection. As a rule it is not desirable to specify a particular plane as the plane of section; yet, as it is convenient to be able to use the nomenclature of plane geometry, we alway suppose such a section made. The outcome of these considerations is that in the plane figure lines (or curves) are given by one, points by two, homogeneous equations in six coordinates x_r subject to the relations $\Sigma (x_r) \equiv 0$, $\Sigma (X_r^2 x_r) \equiv 0$, and, further, the equations are of such a form that the substitution of $x_r + \lambda X_r$ for x_r does not alter them. For instance we speak of the equation of the plane through K and the line ce of the figure in space, found a short way back, as being the equation of the line ce of the projected plane figure of Pascal's Hexagram.

The verification of Veronese's theorems concerning the Pascal Hexagram by means of these equations is usually instantaneous and in no case presents any difficulty, but there is no reason to consider the theorems in detail. Cremona realised that the whole series of propositions were in truth only the relics of the simpler properties of a three-dimensional figure, and we have gone further in connecting them with four-dimensional space. To quote the theorems one by one is wearisome; but to be able to describe the properties of a set of coplanar lines by the phrase Veronese's properties of Pascal's Hexagram is so convenient for my purpose that some consideration of the meaning and origin of the phrase was called for. That all the properties (α) (β) (γ) (δ) (ϵ) (ζ) of II. do hold when a, b, c, d, e, f denote six points on a conic will in future be taken as proved by the foregoing investigation. It may be said that the equations used by Cremona in his paper on Pascal's Hexagram are very inconvenient, while those which we have used, discovered also by Cremona at a later time but not applied to this subject, are perfectly symmetrical. [A system of equations in u coordinates may also be employed for this case, and there is no reason why each kind of coordinate should not be used both here and in the general case of projections of space-sections of a Hexastigm. For if loci in the S_4 be first cut by a space whose coordinates are U_r and then projected from a point of this space whose coordinates are X_r, the resulting plane loci will be defined by equations in coordinates x_r or u_r subject to conditions

$$\Sigma(x_r) \equiv 0;\ \Sigma(u_r x_r) \equiv 0;\ \Sigma(u_r) \equiv 0;$$
$$\Sigma(U_r x_r) \equiv 0;\ \Sigma(X_r u_r) \equiv 0;$$
$$\Sigma(X_r) = 0;\ \Sigma(U_r X_r) = 0;\ \Sigma(U_r) = 0:$$

the equations being always of such a nature that a substitution of $x_r + \lambda X_r$ for x_r or of $u_r + \mu U_r$ for u_r does not affect them].

Section V.

Generalization of these results.

First Method.

In the course of the last section it was observed that the space-sections of the transversal planes of a six-space system in S_4 were a set of lines already familiar to mathematicians in connexion with surfaces of the third order; they form in the most general case such a set of lines as remains when from the twenty-seven lines of the surface we reject a double-six. The properties of these lines in space which are consequences of the nature of the Hexastigm, prove to be well known deductions from this new mode of defining them and need not detain us; the equations which we obtained for them from the Hexastigm were obtained by Cremona from three-dimensional considerations. The plane systems of fifteen lines which possess the properties (α), (β), (γ), (δ), (ϵ), (ζ) of II., and all the rest of Veronese's properties of Pascal's Hexagram, are then projections of certain sets of lines of a cubic surface and have been discussed by Geiser, (*Math. Ann.* I. p. 129).

When a cubic surface is given, and lines drawn through a point K to touch the surface, their points of contact lie on a quadric surface, the polar of K; each line that lies on the cubic surface meets this quadric in two points and therefore touches in two distinct points the cone with vertex K that envelopes the cubic surface. The projections from centre K of the lines which lie on a cubic surface are double tangents of the section of the cone with vertex K which envelopes the surface. When the cubic surface is a space-section of Segre's cubic variety V_3, the cone is a space-section of the cone in S_4 formed of lines which pass through K and touch V_3; if (X_r) be the coordinates of K, the equation of the cone is the discriminant of

$$\Sigma (x_r + \lambda X_r)^3 = 0,$$

and is in general of the sixth order. If the equation of the cone be required, a considerable simplification is effected by imposing the condition $\Sigma (X_r^2 x_r) \equiv 0$, upon the coordinates x, *i.e.* by projecting upon the polar plane of K. But we may state at once that, if K is not on the cubic surface, the section of the enveloping cone is a sextic curve of a particular type, distinguished by its having six cusps which lie on a conic. Since the cubic surface has thirty-six double-sixes of lines upon it, we infer that, from the twenty-seven double tangents which Plücker's equations shew this curve to possess, we may select thirty-six sets of fifteen lines which possess all Veronese's properties of Pascal's Hexagram. I do not know of any discussion of the properties of this sextic curve; its interest appears to be due wholly to its relation to the surface of the third order.

A far more important series of results springs from the particular case when K lies on the cubic surface. We cannot now choose the polar plane of K as the plane on which we project, for it now passes through K; it is best not to specify any plane for the purpose. The tangent cone from K is now of the fourth order, and if, as before (X_r) be the coordinates of K, and (U_r), those of the space by which the four-dimensional figure is cut, we have the following system of equations for the enveloping cone:—

$$3 \{\Sigma (X_r x_r^2)\}^2 = 4 \{\Sigma (x_r^3)\} \{\Sigma (X_r^2 x_r)\};$$

$$\Sigma (x_r) \equiv 0; \; \Sigma (U_r x_r) \equiv 0; \; \Sigma (X_r) = 0; \; \Sigma (U_r X_r) = 0;$$

$$\Sigma (X_r^3) = 0.$$

In the special case when K lies on the cubic surface, the section of the cone with vertex K which envelopes the surface is a curve of the fourth order without singularities, twenty-seven of whose double tangents are projections from vertex K of the lines of the cubic surface, the remaining double tangent being the intersection of the tangent plane at K with the plane of the curve. Now the system of double tangents of a quartic curve has been widely studied, and receives very thorough treatment from a point of view suited to the present geometrical investigation in Salmon's *Higher Plane Curves.* It is there shewn that the twenty eight double tangents may be denoted by the pairs of eight symbols a, b, c, d, e, f, g, h; (Salmon uses 1, 2, 3, 4, 5, 6, 7, 8); but the complete symmetry of the system is not fully shewn by this notation. A rule has been given by Cayley, founded on Hesse's investigations, called the rule of the *bifid substitution*, which removes this defect. The simplest of many possible ways of connecting the notations for the lines of the cubic surface which we have used and that just explained for double tangents of the quadric is to denote the double tangent derived from the tangent plane at K by gh; those derived by projection of members of a double six by ag, bg, eg, dg, eg, fg; ah, bh, ch, dh, eh, fh respectively. The remaining fifteen double tangents, (which form a set derivable by projection from the space-section of the transversal planes of a six-space system) are represented by the same symbols associated with each in Table I. Thus we arrive at a theorem concerning double tangents of a plane quartic which may be stated in the following curious form:—

The fifteen double tangents of a plane curve of the fourth order, denoted in Hesse's Algorithm by symbols formed of pairs of six symbols a, b, c, d, e, f, *possess all the properties of the Pascal Hexagram formed by lines,* (*naturally represented by the same symbols*), *which join each two of six points* a, b, c, d, e, f, *of a conic section.* That some of these properties should have been discovered independently is not to be wondered at: Salmon quotes (p. 234) two sets of six double tangents studied by Steiner and Hesse, the former set of which the six ab, bc, ca, de, ef, fd, are typical touch a conic, as we saw in (β); the latter set, ab, bc, cd, de, ef, fa, have their intersections on a line, which we call a Pascal line. We find ourselves in possession of an immense extension of Steiner's and Hesse's results, and have also a much clearer view of the inner principle on which these results rest, than can be obtained by slowly developing elementary geometrical properties of the lines.

A remarkable fact, not however without parallel, comes to light when we seek, by aid of the rule of the bifid substitution, for a distinctive geometrical property of such a set of fifteen double tangents. Selecting any pair of double tangents (ag

and *ah* for example), we find five other pairs (*bg*, *bh*; *cg*, *ch*; *dg*, *dh*; *eg*, *eh*; *fg*, *fh*), such that the eight contacts of any two of the six pairs lie on a conic: such a system of six pairs may be chosen in sixty-three ways. Of the remain double tangents, *any* fifteen possess all the properties of the Hexagram. Thus whereas, in the case of Pascal's Hexagram or of the double tangents of the sextic curve above described, we have to deal with sets of fifteen lines which possess a long series of properties on account of a quite definite cause, we here find the fifteen lines joined by a sixteenth, which forms with them an absolutely symmetrical family, any fifteen of whose members possess all the properties of the former sets. Our sense of symmetry alone shews the necessity for considering sets of sixteen double tangents of the quartic rather than fifteen; but the discussion may be postponed. It will be seen that the statement that the fifteen double tangents of a quartic curve *ab*, *ac*,...*ef*, possess all Veronese's properties of Pascal's Hexagram, does not include all their properties: the statement in fact deals which only forty-five of their intersections and ignores the remaining sixty, which are of equal importance in the case of the quartic curve, but coalesce by tens in the Hexagram. For instance the rule of the bifid substitution shows that in the case of a quartic curve the points of intersection of *ab* with *ac*, of *ad* with *ae*, and of *bc* with *de*, are collinear: the same theorem is nugatory in the Hexagram and does not hold in the case of the sextic curve.

The relations between the different families of fifteen double tangents of the sextic or of sixteen double tangents of the quartic; how far the Pascal lines, Kirkman points, Steiner points, etc. of different families are common, and so forth, will be passed over entirely. Between the very special case of Pascal's Hexagram and the general quartic (or the above mentioned sextic) are numerous other special cases; for example, if the quartic have a node, the properties of the Hexagram belong to any fifteen of the sixteen double tangents, and in a modified form to certain sets of lines composed partly of double tangents and partly of tangents from the node.

Second Method.

The two operations (section and projection), by means of which the plane families of lines just considered were derived from the transversal planes, may be taken in the reverse order without altering the final result; the intermediate stage, the figure in three-dimensions through which we pass, will be quite different from the former. But, in place of applying the process of projection followed by that of section to the six-space system, it is more convenient to turn to the reciprocal problem, viz., that of deriving a set of fifteen points in a plane from the transversal lines of a six-point system by first taking a space-section of the figure and then projecting on a plane. The resulting family of points will necessarily possess properties reciprocal to Veronese's properties of Pascal's Hexagram, and will also be necessarily reciprocal to a family of lines such as we have just been discussing; the interest of this second method lies in the intermediate stage, which is of quite a new character, not in the final stage, which is bound to be simply reciprocal to that of the earlier method. One advantage of arranging the two methods in this manner is that we may follow the two simultaneously, by taking a space-section of the complete Hexastigm, (which comprises both a six-space and a six-point system), and then projecting the complete section on a plane: the space-section will thus include fifteen lines derived from the transversal planes of the six-space system, and fifteen points derived from the transversal lines of the six-point system. What is the nature of these fifteen points? Do they form a configuration already known? We have seen that they lie by sixes in ten cardinal planes, and we have to some extent discussed their properties under the title principal points in II. A better clue for the purpose of connecting them with known results is furnished by the variety reciprocal to V_3 which is associated with the six-point system in the same manner that V_3 is associated with the six-space system. The reciprocal of a cubic variety with ten double points is of order $3.2^3-2.10=4$, and we therefore denote it by V_4.

In the coordinates u the equation of V_4 is $\Sigma(u_r^3)=0$, and therefore in coordinates x the equation is found by eliminating λ and the quantities u_r from

$$x_r+\lambda=u_r^2;\ \Sigma(u_r)\equiv 0,\ \Sigma(u_r^3)=0;\ (r=1,2,3,4,5,6),$$

and, if advisable, using $\Sigma(x_r)\equiv 0$ to simplify the result: the quantities u_r being thus roots of a sextic equation in some variable θ lacking terms in θ^5 and θ^3; the quantities x_r are roots of an equation which differs from a perfect square only in its two last terms: from this we deduce that

$$\{\Sigma(x_r^2)\}^2=4\Sigma(x_r^4);\quad \Sigma(x_r)\equiv 0;$$

but many other forms may be given to the result. The variety V_4 is of the fourth order, has each transversal of the six-point system as a double line and each cardinal space as a singular tangent space, *i.e.* is cut by each cardinal space in a quadric surface taken twice. A space-section of V_4 is therefore a quartic surface which has fifteen double points and ten singular tangent planes, the principal points and cardinal planes of the space-section of the six-point system.

Conversely, if a quartic surface have fifteen double points, it may be shown that it must be a space-section of a quartic variety such as V_4; it does not seem necessary to give the proof; incidentally we notice some other forms to which the equation of V_4 may be reduced, such as

$$\{(x_1-x_4)^2-(x_2+x_5-x_3-x_6)^2\}^{\frac{1}{2}}+\{(x_2-x_5)^2-(x_3+x_6-x_1-x_4)^2\}^{\frac{1}{2}} + \{(x_3-x_6)^2-(x_1+x_4-x_2-x_5)^2\}^{\frac{1}{2}};$$

or again the equation of V_4 is the discriminant of

$$(\lambda+x_2+x_3-x_1)(\lambda+x_3+x_1-x_2)(\lambda+x_1+x_2-x_3) - (\lambda+x_5+x_6-x_4)(\lambda+x_6+x_4-x_5)(\lambda+x_4+x_5-x_6).$$

It is not, however, my purpose to develop properties of V_4, except in so far as they throw light on the families of lines and points we have discovered. What concerns us at present is that we have obtained a second quite new way of arriving at the plane families of lines, or rather the families of points derived by the principle of duality; in fact we may assert:—

The projections on a plane of the fifteen double points of a quartic surface form a family of points possessing properties reciprocal to Veronese's series of properties of Pascal's Hexagram.

Of the two methods the latter is to be preferred; the fifteen principal points of a space-section of the six-point system are determined by ten cardinal planes which form a figure in space quite readily pictured mentally if we conceive the planes to be disposed as in Figure 2. The method of the fifteen lines which lie by threes in fifteen planes, as Cremona expressed it, or of fifteen lines of a cubic surface excluding a double six, which formed the intermediate stage in the first method are by no means so easy to realize, even after a model has been studied. As to the two-dimensional figure there is nothing to choose: in fact it becomes more and more apparent that the plane figure must be considered simply as a projection of a space-figure, and its properties thus derived; any attempt to think of the plane figure by itself, purely as a two-dimensional system, entangles us in a maze of elementary theorems absolutely bewildering in their numbers. The space-figures are not so complex as to confuse us and can be realized with a slight effort; if the four-dimensional figure could be pictured mentally, to discuss even the space-figures would be superfluous. It might be thought that the fact that the first method depended on a cubic surface and the second on a quartic told in favour of the former; but even this appears to me to be untrue, for these quartic surfaces are of particular interest.

An important special case presents itself when the space of section touches V_4 at some point K; for the resulting surface must then have another node at K or sixteen in all. The surface is in fact the much-studied Kummer's surface; not only are the fifteen nodes ab, ac, ..., ef joined by a new node K, but the ten singular tangent planes are joined by six others which pass through K, reciprocals of the six lines which were proved to pass through each point of Segre's variety V_3 and lie on the surface. These six lines were denoted by a, b, c, d, e, f in such a way that a met the five planes ab, ac, ad, ae, af in the six-space system; if the six planes which pass through K, reciprocal to these lines, be here denoted by a, b, c, d, e, f, we find ourselves making use of the ordinary notation for a Kummer's configuration (see Reye, *Geometrie der Lage*, latest edition; or Sturm, *Liniengeometrie*, Vol. 2) viz.

(1) K, one of the nodal points;

(2) a, b, c, d, e, f the six planes which pass through K;

(3) ab, ac, ..., ef, the other fifteen points, so named that ab lies on the planes a and b.

(4) $C\,(abc.def)$, or simply $abc.def$, the remaining ten planes, each containing six of the points (3).

The fifteen points (3) and the ten planes (4) retain all their previous properties, but also acquire some new ones, *e.g.* that ab, ac, ad, ae, af are now coplanar; but *any* fifteen of the sixteen nodes possess all the properties proved for these fifteen; any fifteen, for example, have sixty Pascal points, &c., not, however, all of them distinct, for it appears that there are in all only two hundred and forty Pascal points; each in fact belongs to four different sets of fifteen points. We observe also that, by taking the vertex of projection at K, the projections of the fifteen points ab, ac, ..., ef are intersections of six tangents to a conic; and it is clear that, as Kummer's surface is self-dualistic, we may derive, by cutting its singular planes by a plane, a set of sixteen lines, which has all the properties of the lines reciprocal to the projections of its nodes.

Each method thus defines the most general family of lines which possess Veronese's properties of the Pascal Hexagram, or the reciprocal family of points, by means of double tangents of a sextic curve, or by projection of the nodes of a surface of the sixth class, (for the class of a quartic with fifteen nodes is 6): each shows the existence of an important special case of double tangents of a quartic when the phenomenon of the appearance of a new member of the system occurs; and finally there is the case of Pascal's Hexagram, or its reciprocal, in which the plane figure is so approached that the new member of the family is made indeterminate (by choice of a centre of projection coinciding with it, or some such means). The case of the quartic curve obviously demands further study as regards the mutual relations of the Pascal lines, Kirkman points, Steiner points, &c., &c., of the different sets of fifteen lines: the sets of sixteen double tangents in question are represented either by gh and ab, ac, ..., ef as before, or by one of four symbols a, b, c, d associated with one of the four e, f, g, h. It is, however, clear that the second method of passing from four to two dimensions enables us to discuss the matter in connexion with the comparatively simple three-dimensional figure instead of the complex plane figure. For the fifteen nodes and ten singular planes of a quartic surface which has fifteen nodes may be investigated by means of a perfectly symmetrical set of symbols and equations; their properties are directly connected with the plane figure on the one hand and with the simpler four-dimensional figure on the other, and may be developed with a very slight amount of labour. The effect of the sixteenth node on these and the symmetry of the system is better dealt with in space than in the plane; at the same time it does not depend upon space of four dimensions; (except in so far as the symmetrical system of equations for the quartic with fifteen nodes was suggested by considerations of an S_4); and so does not fall within the range of this paper.

That any tangent plane of a cubic surface, and the twenty-seven planes through the point of contact and the twenty-seven lines of the surface should form an absolutely symmetrical set of planes was once pointed out to me by Professor Cayley as a remarkable fact which must imply a series of quite unknown properties of the cubic surface. The same interesting fact appears in Segre's cubic variety, but I can suggest no explanation.

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